10.4 Exercises

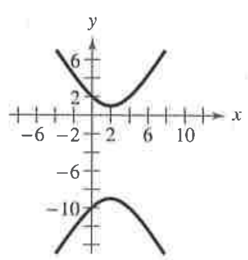
Vocabulary: Fill in the blanks.

1. A ________ is the set of all points (x, y) in a plane for which the absolute value of the difference of the distances from two distinct fixed points, called ________, is constant.
2. The graph of a hyperbola has two disconnected parts called ________.
3. The line segment connecting the vertices of a hyperbola is the ________ ________, and its midpoint is the ________ of the hyperbola.
4. Every hyperbola has two ________ that intersect at the center of the hyperbola.

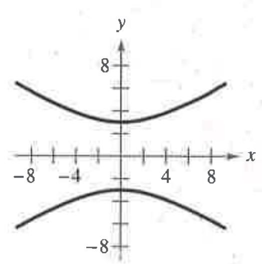
Skills and Applications

Matching In Exercises 5–8, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]

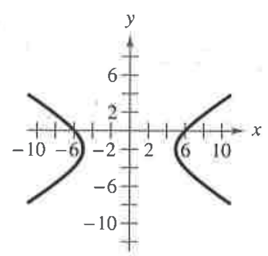
(a)



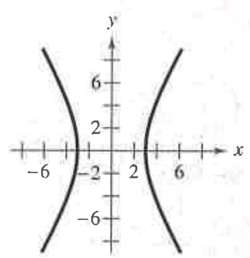
(b)



(c)



(d)



5. $\frac{y^2}{9} - \frac{x^2}{25} = 1$
6. $\frac{x^2}{9} - \frac{y^2}{25} = 1$
7. $\frac{x^2}{25} - \frac{(y+2)^2}{9} = 1$
8. $\frac{(y+4)^2}{25} - \frac{(x-2)^2}{9} = 1$

Finding the Standard Equation of a Hyperbola In Exercises 9–18, find the standard form of the equation of the hyperbola with the given characteristics.

9. Vertices: $(0, \pm 2)$; foci: $(0, \pm 4)$
10. Vertices: $(\pm 4, 0)$; foci: $(\pm 6, 0)$
11. Vertices: $(2, 0)$, $(6, 0)$; foci: $(0, 0)$, $(8, 0)$
12. Vertices: $(2, 3)$, $(2, -3)$; foci: $(2, 6)$, $(2, -6)$
13. Vertices: $(4, 1)$, $(4, 9)$; foci: $(4, 0)$, $(4, 10)$
14. Vertices: $(-1, 1)$, $(3, 1)$; foci: $(-2, 1)$, $(4, 1)$
15. Vertices: $(2, 3)$, $(2, -3)$; passes through the point $(0, 5)$
16. Vertices: $(-2, 1)$, $(2, 1)$; passes through the point $(5, 4)$
17. Vertices: $(0, -3)$, $(4, -3)$; passes through the point $(-4, 5)$
18. Vertices: $(1, -3)$, $(1, -7)$; passes through the point $(5, -11)$

Sketching a Hyperbola In Exercises 19–32, find the center, vertices, foci, and the equations of the asymptotes of the hyperbola. Then sketch the hyperbola using the asymptotes as an aid.

19. $x^2 - y^2 = 1$
20. $\frac{x^2}{9} - \frac{y^2}{16} = 1$
21. $\frac{1}{36}y^2 - \frac{1}{100}x^2 = 1$
22. $\frac{1}{144}x^2 - \frac{1}{169}y^2 = 1$
23. $2y^2 - \frac{x^2}{2} = 2$
24. $\frac{y^2}{3} - 3x^2 = 3$
25. $\frac{(x-1)^2}{4} - \frac{(y+2)^2}{1} = 1$
26. $\frac{(x+3)^2}{144} - \frac{(y-2)^2}{25} = 1$
27. $\frac{(y+6)^2}{1/9} - \frac{(x-2)^2}{1/4} = 1$
28. $\frac{(y-1)^2}{1/4} - \frac{(x+3)^2}{1/16} = 1$
29. $9x^2 - y^2 - 36x - 6y + 18 = 0$
30. $x^2 - 9y^2 + 36y - 72 = 0$
31. $4x^2 - y^2 + 8x + 2y - 1 = 0$
32. $16y^2 - x^2 + 2x + 64y + 64 = 0$

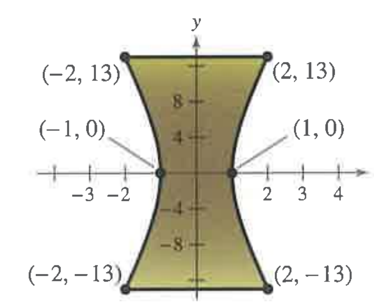
Graphing a Hyperbola In Exercises 33–38, use a graphing utility to graph the hyperbola and its asymptotes. Find the center, vertices, and foci.

33. $2x^2 - 3y^2 = 6$
34. $6y^2 - 3x^2 = 18$
35. $25y^2 - 9x^2 = 225$
36. $25x^2 - 4y^2 = 100$
37. $9y^2 - x^2 + 2x + 54y + 62 = 0$
38. $9x^2 - y^2 + 54x + 10y + 55 = 0$

Finding the Standard Equation of a Hyperbola In Exercises 39–48, find the standard form of the equation of the hyperbola with the given characteristics.

39. Vertices: $(\pm 1, 0)$; asymptotes: $y = \pm 5x$
40. Vertices: $(0, \pm 3)$; asymptotes: $y = \pm 3x$
41. Foci: $(0, \pm 8)$; asymptotes: $y = \pm 4x$
42. Foci: $(\pm 10, 0)$; asymptotes: $y = \pm \frac{3}{4}x$
43. Vertices: $(1, 2)$, $(3, 2)$; asymptotes: $y = x$, $y = 4 - x$
44. Vertices: $(3, 0)$, $(3, 6)$; asymptotes: $y = 6 - x$, $y = x$
45. Vertices: $(3, 0)$, $(3, 4)$; asymptotes: $y = \frac{2}{3}x$, $y = 4 - \frac{2}{3}x$
46. Vertices: $(-4, 1)$, $(0, 1)$; asymptotes: $y = x + 3$, $y = -x - 1$
47. Foci: $(-1, -1)$, $(9, -1)$; asymptotes: $y = \frac{3}{4}x - 4$, $y = -\frac{3}{4}x + 2$
48. Foci: $(9, \pm 2\sqrt{10})$; asymptotes: $y = 3x - 27$, $y = -3x + 27$

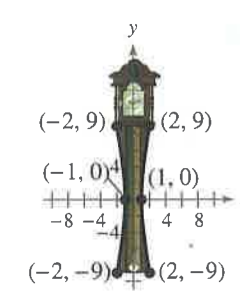
49. **Art** A cross section of a sculpture can be modeled by a hyperbola (see figure).



(a) Write an equation that models the curved sides of the sculpture.

(b) Each unit in the coordinate plane represents 1 foot. Find the width of the sculpture at a height of 18 feet.

50. **Clock** The base of a clock has the shape of a hyperbola (see figure).



(a) Write an equation of the cross section of the base.

(b) Each unit in the coordinate plane represents $\frac{1}{2}$ foot. Find the width of the base 4 inches from the bottom.

51. **Sound Location** You and a friend live 4 miles apart. You hear a clap of thunder from lightning 18 seconds before your friend hears it. Where did the lightning occur? (Assume sound travels at 1100 feet per second.)

52. **Sound Location** Listening station A and listening station B are located at $(3300, 0)$ and $(-3300, 0)$, respectively. Station A detects an explosion 4 seconds before station B. (Assume the coordinate system is measured in feet and sound travels at 1100 feet per second.)

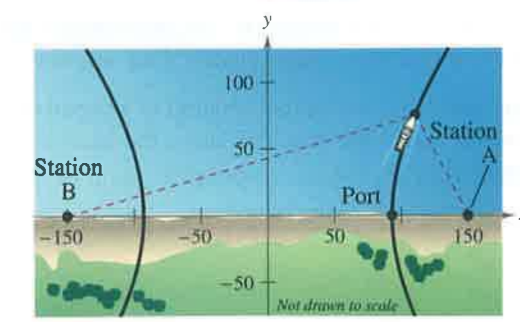
(a) Where did the explosion occur?

(b) Station C is located at $(3300, 1100)$ and detects the explosion 1 second after station A. Find the coordinates of the explosion.

53. **Navigation**

Long-distance radio navigation for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci.

Assume that two stations 300 miles apart are positioned on a rectangular coordinate system with coordinates $(-150, 0)$ and $(150, 0)$ and that a ship is traveling on a hyperbolic path with coordinates $(x, 75)$ (see figure).

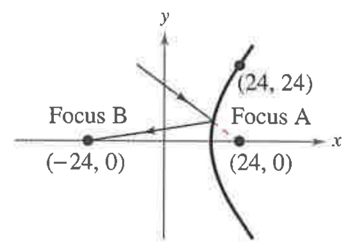


(a) Find the x-coordinate of the position of the ship when the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).

(b) Determine the distance between the port and station A.

(c) Find a linear equation that approximates the ship's path as it travels far away from the shore.

54. Hyperbolic Mirror A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at focus A is reflected to focus B (see figure). Find the vertex of the mirror when its mount at the top edge of the mirror has coordinates (24, 24).



Classifying a Conic from a General Equation **In Exercises 55–66, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.**

55. $9x^2 + 4y^2 - 18x + 16y - 119 = 0$
56. $x^2 + y^2 - 4x - 6y - 23 = 0$
57. $4x^2 - y^2 - 4x - 3 = 0$
58. $y^2 - 6y - 4x + 21 = 0$
59. $y^2 - 4x^2 + 4x - 2y - 4 = 0$
60. $y^2 + 12x + 4y + 28 = 0$
61. $4x^2 + 25y^2 + 16x + 250y + 541 = 0$
62. $4y^2 - 2x^2 - 4y - 8x - 15 = 0$
63. $25x^2 - 10x - 200y - 119 = 0$
64. $4y^2 + 4x^2 - 24x + 35 = 0$
65. $100x^2 + 100y^2 - 100x + 400y + 409 = 0$
66. $9x^2 + 4y^2 - 90x + 8y + 228 = 0$

Exploration

True or False? **In Exercises 67–69, determine whether the statement is true or false. Justify your answer.**

67. In the standard form of the equation of a hyperbola, the larger the ratio of b to a, the larger the eccentricity of the hyperbola.

68. If the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $a, b > 0$, intersect at right angles, then $a = b$.

69. The graph of

$$x^2 - y^2 + 4x - 4y = 0$$

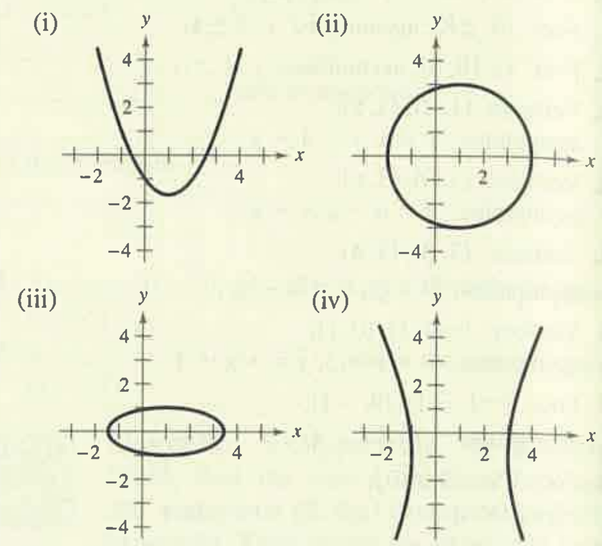
is a hyperbola.

70. Think About It Write an equation whose graph is the bottom half of the hyperbola

$$9x^2 - 54x - 4y^2 + 8y + 41 = 0.$$

71. Writing Explain how to use a rectangle to sketch the asymptotes of a hyperbola.

72. HOW DO YOU SEE IT? Match each equation with its graph.



(a) $4x^2 - y^2 - 8x - 2y - 13 = 0$
(b) $x^2 + y^2 - 2x - 8 = 0$
(c) $2x^2 - 4x - 3y - 3 = 0$
(d) $x^2 + 6y^2 - 2x - 5 = 0$

73. Error Analysis Describe the error in finding the asymptotes of the hyperbola

$$\frac{(y+5)^2}{9} - \frac{(x-3)^2}{4} = 1.$$

$$y = k \pm \frac{b}{a}(x - h)$$
$$= -5 \pm \tfrac{2}{3}(x - 3)$$

The asymptotes are $y = \frac{2}{3}x - 7$ and $y = -\frac{2}{3}x - 3$.

74. Think About It Consider a hyperbola centered at the origin with a horizontal transverse axis. Use the definition of a hyperbola to derive its standard form.

75. Points of Intersection Sketch the circle $x^2 + y^2 = 4$. Then find the values of C so that the parabola $y = x^2 + C$ intersects the circle at the given number of points.

(a) 0 points
(b) 1 point
(c) 2 points
(d) 3 points
(e) 4 points

10.5 Rotation of Conics

Rotated conics can model objects in real life. For example, in Exercise 63 on page 734, you will use a rotated parabola to model the cross section of a satellite dish.

- **Rotate the coordinate axes to eliminate the *xy*-term in equations of conics.**
- **Use the discriminant to classify conics.**

Rotation

In the preceding section, you classified conics whose equations were written in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

The graphs of such conics have axes that are parallel to one of the coordinate axes. Conics whose axes are rotated so that they are not parallel to either the x-axis or the y-axis have general equations that contain an xy-term.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad \text{Equation in } xy\text{-plane}$$

To eliminate this xy-term, use a procedure called **rotation of axes.** The objective is to rotate the x- and y-axes until they are parallel to the axes of the conic. The rotated axes are denoted as the x'-axis and the y'-axis, as shown in Figure 10.31. After the rotation, the equation of the conic in the $x'y'$-plane will have the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0. \qquad \text{Equation in } x'y'\text{-plane}$$

This equation has no $x'y'$-term, so you can obtain a standard form by completing the square. The theorem below identifies how much to rotate the axes to eliminate the xy-term and also the equations for determining the new coefficients A', C', D', E', and F'.

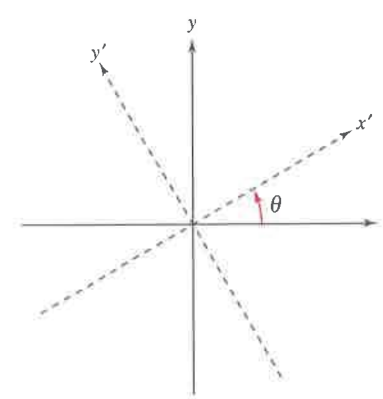


Figure 10.31

Rotation of Axes to Eliminate an *xy*-Term

The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where $B \neq 0$, can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

by rotating the coordinate axes through an angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

The coefficients of the new equation are obtained by making the substitutions

$$x = x'\cos\theta - y'\sin\theta$$

and

$$y = x'\sin\theta + y'\cos\theta.$$

Remember that the substitutions

$$x = x'\cos\theta - y'\sin\theta$$

and

$$y = x'\sin\theta + y'\cos\theta$$

should eliminate the $x'y'$-term in the rotated system. Use this as a check of your work. If you obtain an equation of a conic in the $x'y'$-plane that contains an $x'y'$-term, you know that you have made a mistake.