A **cycloid** is a curve traced by a point P on a circle as the circle rolls along a straight line in a plane.

EXAMPLE 5 **Parametric Equations for a Cycloid**

Write parametric equations for a cycloid traced by a point P on a circle of radius a units as the circle rolls along the x-axis given that P is at a minimum when $x = 0$.

Solution Let the parameter θ be the measure of the circle's rotation, and let the point $P(x, y)$ begin at the origin. When $\theta = 0$, P is at the origin; when $\theta = \pi$, P is at a maximum point $(\pi a, 2a)$; and when $\theta = 2\pi$, P is back on the x-axis at $(2\pi a, 0)$. From the figure below, $\angle APC = \pi - \theta$. So, you have

$$\sin\theta = \sin(\pi - \theta) = \sin(\angle APC) = \frac{AC}{a} = \frac{BD}{a}$$

$$\cos\theta = -\cos(\pi - \theta) = -\cos(\angle APC) = -\frac{AP}{a}$$

which implies that $BD = a\sin\theta$ and $AP = -a\cos\theta$. The circle rolls along the x-axis, so you know that $OD = \widehat{PD} = a\theta$. Furthermore, $BA = DC = a$, so you have

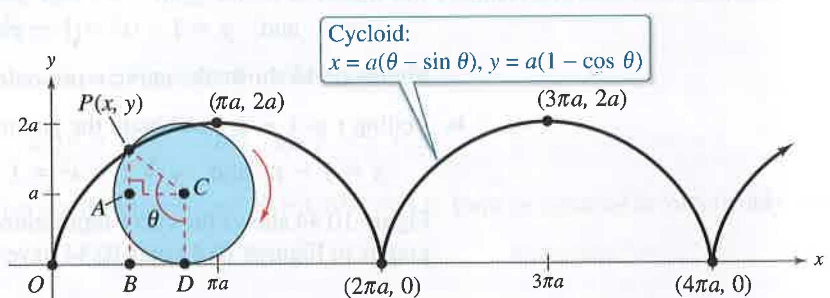
$$x = OD - BD = a\theta - a\sin\theta$$

and

$$y = BA + AP = a - a\cos\theta.$$

The parametric equations are $x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$.

REMARK In Example 5, $\widehat{PD}$ represents the arc of the circle between points P and D.



TECHNOLOGY Use a graphing utility in *parametric* mode to obtain a graph similar to the one in Example 5 by graphing

$X_{1T} = T - \sin T$

and

$Y_{1T} = 1 - \cos T$.

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Write parametric equations for a cycloid traced by a point P on a circle of radius a as the circle rolls along the x-axis given that P is at a maximum when $x = 0$.

Summarize (Section 10.6)

1. Explain how to evaluate a set of parametric equations for given values of the parameter and sketch a curve represented by a set of parametric equations *(pages 735 and 736)*. For an example of sketching a curve represented by a set of parametric equations, see Example 1.
2. Explain how to rewrite a set of parametric equations as a single rectangular equation by eliminating the parameter *(page 737)*. For examples of sketching curves by eliminating the parameter, see Examples 2 and 3.
3. Explain how to find a set of parametric equations for a graph *(page 739)*. For examples of finding sets of parametric equations for graphs, see Examples 4 and 5.

10.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: **Fill in the blanks.**

1. If f and g are continuous functions of t on an interval I, then the set of ordered pairs $(f(t), g(t))$ is a ________ C.
2. The ______ of a curve is the direction in which the curve is traced for increasing values of the parameter.
3. The process of converting a set of parametric equations to a corresponding rectangular equation is called ______ the ______.
4. A curve traced by a point on the circumference of a circle as the circle rolls along a straight line in a plane is a ______.

Skills and Applications

5. **Sketching a Curve** Consider the parametric equations $x = \sqrt{t}$ and $y = 3 - t$.
   (a) Create a table of x- and y-values using $t = 0, 1, 2, 3$, and 4.
   (b) Plot the points (x, y) generated in part (a), and sketch a graph of the parametric equations.
   (c) Sketch the graph of $y = 3 - x^2$. How do the graphs differ?
6. **Sketching a Curve** Consider the parametric equations $x = 4\cos^2\theta$ and $y = 2\sin\theta$.
   (a) Create a table of x- and y-values using $\theta = -\pi/2, -\pi/4, 0, \pi/4$, and $\pi/2$.
   (b) Plot the points (x, y) generated in part (a), and sketch a graph of the parametric equations.
   (c) Sketch the graph of $x = -y^2 + 4$. How do the graphs differ?

Sketching a Curve **In Exercises 7–12, sketch and describe the orientation of the curve given by the parametric equations.**

7. $x = t, \quad y = -5t$
8. $x = 2t - 1, \quad y = t + 4$
9. $x = t^2, \quad y = 3t$
10. $x = \sqrt{t}, \quad y = 2t - 1$
11. $x = 3\cos\theta, \quad y = 2\sin^2\theta, \quad 0 \le \theta \le \pi$
12. $x = \cos\theta, \quad y = 2\sin\theta, \quad 0 \le \theta \le 2\pi$

Sketching a Curve **In Exercises 13–38, (a) sketch the curve represented by the parametric equations (indicate the orientation of the curve) and (b) eliminate the parameter and write the resulting rectangular equation whose graph represents the curve. Adjust the domain of the rectangular equation, if necessary.**

13. $x = t, \quad y = 4t$
14. $x = t, \quad y = -\frac{1}{2}t$
15. $x = -t + 1, \quad y = -3t$
16. $x = 3 - 2t, \quad y = 2 + 3t$
17. $x = \frac{1}{4}t, \quad y = t^2$
18. $x = t, \quad y = t^3$
19. $x = t^2, \quad y = -2t$
20. $x = -t^2, \quad y = \dfrac{t}{3}$
21. $x = \sqrt{t}, \quad y = 1 - t$
22. $x = \sqrt{t + 2}, \quad y = t - 1$
23. $x = \sqrt{t} - 3, \quad y = t^3$
24. $x = \sqrt{t - 1}, \quad y = \sqrt[3]{t - 1}$
25. $x = t + 1$, $y = \dfrac{t}{t + 1}$
26. $x = t - 1$, $y = \dfrac{t}{t - 1}$
27. $x = 4\cos\theta$, $y = 2\sin\theta$
28. $x = 2\cos\theta$, $y = 3\sin\theta$
29. $x = 1 + \cos\theta$, $y = 1 + 2\sin\theta$
30. $x = 2 + 5\cos\theta$, $y = -6 + 4\sin\theta$
31. $x = 2\sec\theta, \quad y = \tan\theta, \quad \pi/2 \le \theta \le 3\pi/2$
32. $x = 3\cot\theta, \quad y = 4\csc\theta, \quad 0 \le \theta \le \pi$
33. $x = 3\cos\theta$, $y = 3\sin\theta$
34. $x = 6\sin 2\theta$, $y = 6\cos 2\theta$
35. $x = e^t, \quad y = e^{3t}$
36. $x = e^{-t}, \quad y = e^{3t}$
37. $x = t^3, \quad y = 3\ln t$
38. $x = \ln 2t, \quad y = 2t^2$

Graphing a Curve **In Exercises 39–48, use a graphing utility to graph the curve represented by the parametric equations.**

39. $x = t$, $y = \sqrt{t}$
40. $x = t + 1$, $y = \sqrt{2 - t}$
41. $x = 2t$, $y = |t + 1|$
42. $x = |t + 2|$, $y = 3 - t$
43. $x = 4 + 3\cos\theta$, $y = -2 + \sin\theta$
44. $x = 4 + 3\cos\theta$, $y = -2 + 2\sin\theta$
45. $x = 2\csc\theta$, $y = 4\cot\theta$
46. $x = \sec\theta$, $y = \tan\theta$
47. $x = \frac{1}{2}t$, $t = \ln(t^2 + 1)$
48. $x = 10 - 0.01e^t$, $y = 0.4t^2$

Comparing Plane Curves In Exercises 49 and 50, determine how the plane curves differ from each other.

49. (a) $x = t$, $y = 2t + 1$ (b) $x = \cos\theta$, $y = 2\cos\theta + 1$
(c) $x = e^{-t}$, $y = 2e^{-t} + 1$ (d) $x = e^{t}$, $y = 2e^{t} + 1$

50. (a) $x = t$, $y = t^2 - 1$ (b) $x = t^2$, $y = t^4 - 1$
(c) $x = \sin t$, $y = \sin^2 t - 1$ (d) $x = e^t$, $y = e^{2t} - 1$

Eliminating the Parameter In Exercises 51–54, eliminate the parameter and obtain the standard form of the rectangular equation.

51. Line passing through (x_1, y_1) and (x_2, y_2):
$x = x_1 + t(x_2 - x_1)$, $y = y_1 + t(y_2 - y_1)$

52. Circle: $x = h + r\cos\theta$, $y = k + r\sin\theta$

53. Ellipse with horizontal major axis:
$x = h + a\cos\theta$, $y = k + b\sin\theta$

54. Hyperbola with horizontal transverse axis:
$x = h + a\sec\theta$, $y = k + b\tan\theta$

Finding Parametric Equations for a Graph In Exercises 55–62, use the results of Exercises 51–54 to find a set of parametric equations to represent the graph of the line or conic.

55. Line: passes through $(0, 0)$ and $(3, 6)$

56. Line: passes through $(3, 2)$ and $(-6, 3)$

57. Circle: center: $(3, 2)$; radius: 4

58. Circle: center: $(-2, -5)$; radius: 7

59. Ellipse: vertices: $(\pm 5, 0)$; foci: $(\pm 4, 0)$

60. Ellipse: vertices: $(7, 3)$, $(-1, 3)$; foci: $(5, 3)$, $(1, 3)$

61. Hyperbola: vertices: $(1, 0)$, $(9, 0)$; foci: $(0, 0)$, $(10, 0)$

62. Hyperbola: vertices: $(4, 1)$, $(8, 1)$; foci: $(2, 1)$, $(10, 1)$

Finding Parametric Equations for a Graph In Exercises 63–66, use the results of Exercises 51 and 54 to find a set of parametric equations to represent the section of the graph of the line or conic. (*Hint:* Adjust the domain of the standard form of the rectangular equation to determine the appropriate interval for the parameter.)

63. Line segment between $(0, 0)$ and $(-5, 2)$

64. Line segment between $(1, -4)$ and $(9, 0)$

65. Left branch of the hyperbola with vertices $(\pm 3, 0)$ and foci $(\pm 5, 0)$

66. Right branch of the hyperbola with vertices $(-4, 3)$ and $(6, 3)$ and foci $(-12, 3)$ and $(14, 3)$

Finding Parametric Equations for a Graph In Exercises 67–78, find a set of parametric equations to represent the graph of the rectangular equation using (a) $t = x$ and (b) $t = 2 - x$.

67. $y = 3x - 2$ **68.** $y = 2 - x$

69. $x = 2y + 1$ **70.** $x = 3y - 2$

71. $y = x^2 + 1$ **72.** $y = 6x^2 - 5$

73. $y = 1 - 2x^2$ **74.** $y = 2 - 5x^2$

75. $y = \dfrac{1}{x}$ **76.** $y = \dfrac{1}{2x}$

77. $y = e^x$ **78.** $y = e^{2x}$

Graphing a Curve In Exercises 79–86, use a graphing utility to graph the curve represented by the parametric equations.

79. Cycloid: $x = 4(\theta - \sin\theta)$, $y = 4(1 - \cos\theta)$

80. Cycloid: $x = \theta + \sin\theta$, $y = 1 - \cos\theta$

81. Prolate cycloid: $x = 2\theta - 4\sin\theta$, $y = 2 - 4\cos\theta$

82. Epicycloid: $x = 8\cos\theta - 2\cos 4\theta$
$y = 8\sin\theta - 2\sin 4\theta$

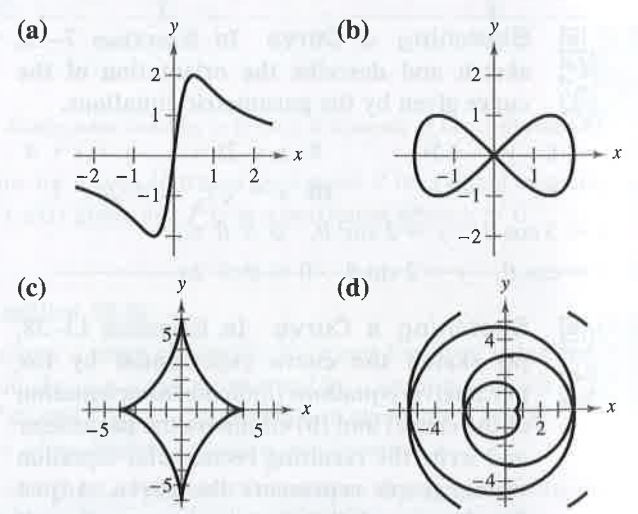
83. Hypocycloid: $x = 3\cos^3\theta$, $y = 3\sin^3\theta$

84. Curtate cycloid: $x = 8\theta - 4\sin\theta$, $y = 8 - 4\cos\theta$

85. Witch of Agnesi: $x = 2\cot\theta$, $y = 2\sin^2\theta$

86. Folium of Descartes: $x = \dfrac{3t}{1 + t^3}$, $y = \dfrac{3t^2}{1 + t^3}$

Matching In Exercises 87–90, match the parametric equations with the correct graph and describe the domain and range. [The graphs are labeled (a)–(d).]



87. Lissajous curve: $x = 2\cos\theta$, $y = \sin 2\theta$

88. Evolute of ellipse: $x = 4\cos^3\theta$, $y = 6\sin^3\theta$

89. Involute of circle: $x = \frac{1}{2}(\cos\theta + \theta\sin\theta)$
$y = \frac{1}{2}(\sin\theta - \theta\cos\theta)$

90. Serpentine curve: $x = \frac{1}{2}\cot\theta$, $y = 4\sin\theta\cos\theta$

Projectile Motion Consider a projectile launched at a height of h feet above the ground at an angle of θ with the horizontal. The initial velocity is v_0 feet per second, and the path of the projectile is modeled by the parametric equations

$$x = (v_0 \cos\theta)t$$

and

$$y = h + (v_0 \sin\theta)t - 16t^2.$$

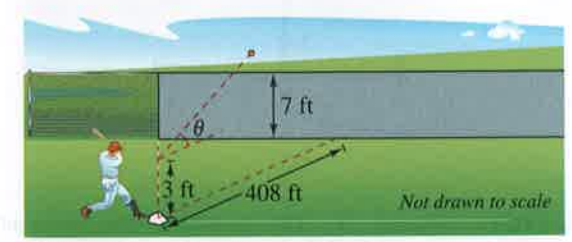
In Exercises 91 and 92, use a graphing utility to graph the paths of a projectile launched from ground level at each value of θ and v_0. For each case, use the graph to approximate the maximum height and the range of the projectile.

91. (a) $\theta = 60°$, $v_0 = 88$ feet per second
(b) $\theta = 60°$, $v_0 = 132$ feet per second
(c) $\theta = 45°$, $v_0 = 88$ feet per second
(d) $\theta = 45°$, $v_0 = 132$ feet per second

92. (a) $\theta = 15°$, $v_0 = 50$ feet per second
(b) $\theta = 15°$, $v_0 = 120$ feet per second
(c) $\theta = 10°$, $v_0 = 50$ feet per second
(d) $\theta = 10°$, $v_0 = 120$ feet per second

93. Path of a Baseball

The center field fence in a baseball stadium is 7 feet high and 408 feet from home plate. A baseball player hits a baseball at a point 3 feet above the ground. The ball leaves the bat at an angle of θ degrees with the horizontal at a speed of 100 miles per hour (see figure).



(a) Write a set of parametric equations that model the path of the baseball. (See Exercises 91 and 92.)

(b) Use a graphing utility to graph the path of the baseball when $\theta = 15°$. Is the hit a home run?

(c) Use the graphing utility to graph the path of the baseball when $\theta = 23°$. Is the hit a home run?

(d) Find the minimum angle required for the hit to be a home run.

94. Path of an Arrow An archer releases an arrow from a bow at a point 5 feet above the ground. The arrow leaves the bow at an angle of 15° with the horizontal and at an initial speed of 225 feet per second.

(a) Write a set of parametric equations that model the path of the arrow. (See Exercises 91 and 92.)

(b) Assuming the ground is level, find the distance the arrow travels before it hits the ground. (Ignore air resistance.)

(c) Use a graphing utility to graph the path of the arrow and approximate its maximum height.

(d) Find the total time the arrow is in the air.

95. Path of a Football A quarterback releases a pass at a height of 7 feet above the playing field, and a receiver catches the football at a height of 4 feet, 30 yards directly downfield. The pass is released at an angle of 35° with the horizontal.

(a) Write a set of parametric equations for the path of the football. (See Exercises 91 and 92.)

(b) Find the speed of the football when it is released.

(c) Use a graphing utility to graph the path of the football and approximate its maximum height.

(d) Find the time the receiver has to position himself after the quarterback releases the football.

96. Projectile Motion Eliminate the parameter t in the parametric equations

$$x = (v_0 \cos\theta)t$$

and

$$y = h + (v_0 \sin\theta)t - 16t^2$$

for the motion of a projectile to show that the rectangular equation is

$$y = -\frac{16\sec^2\theta}{v_0^2}x^2 + (\tan\theta)x + h.$$

97. Path of a Projectile The path of a projectile is given by the rectangular equation

$$y = 7 + x - 0.02x^2.$$

(a) Find the values of h, v_0, and θ. Then write a set of parametric equations that model the path. (See Exercise 96.)

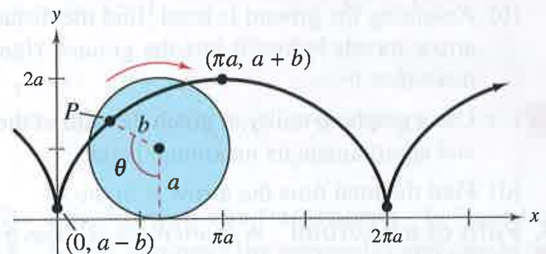
(b) Use a graphing utility to graph the rectangular equation for the path of the projectile. Confirm your answer in part (a) by sketching the curve represented by the parametric equations.

(c) Use the graphing utility to approximate the maximum height of the projectile and its range.

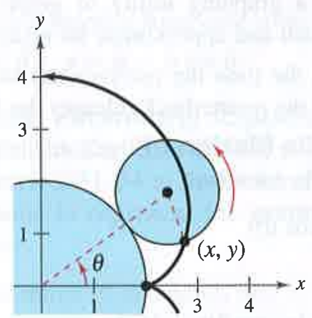
98. Path of a Projectile Repeat Exercise 97 for a projectile with a path given by the rectangular equation

$$y = 6 + x - 0.08x^2.$$

99. Curtate Cycloid A wheel of radius a units rolls along a straight line without slipping. The curve traced by a point P that is b units from the center ($b < a$) is called a **curtate cycloid** (see figure). Use the angle θ shown in the figure to find a set of parametric equations for the curve.



100. Epicycloid A circle of radius one unit rolls around the outside of a circle of radius two units without slipping. The curve traced by a point on the circumference of the smaller circle is called an **epicycloid** (see figure). Use the angle θ shown in the figure to find a set of parametric equations for the curve.



Exploration

True or False? **In Exercises 101–104, determine whether the statement is true or false. Justify your answer.**

101. The two sets of parametric equations

$x = t,\ y = t^2 + 1$ and $x = 3t,\ y = 9t^2 + 1$

correspond to the same rectangular equation.

102. The graphs of the parametric equations

$x = t^2,\ y = t^2$ and $x = t,\ y = t$

both represent the line $y = x$, so they are the same plane curve.

103. If y is a function of t and x is a function of t, then y must be a function of x.

104. The parametric equations

$x = at + h$ and $y = bt + k$

where $a \neq 0$ and $b \neq 0$, represent a circle centered at (h, k) when $a = b$.

105. Writing Write a short paragraph explaining why parametric equations are useful.

106. Writing Explain what is meant by the orientation of a plane curve.

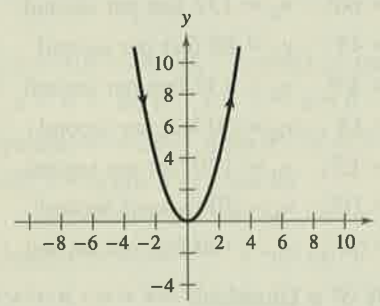
107. Error Analysis Describe the error in finding the rectangular equation for the parametric equations

$x = \sqrt{t-1}$ and $y = 2t$.

$x = \sqrt{t-1} \Rightarrow t = x^2 + 1$

$y = 2(x^2 + 1) = 2x^2 + 2$

108. HOW DO YOU SEE IT? The graph of the parametric equations $x = t$ and $y = t^2$ is shown below. Determine whether the graph would change for each set of parametric equations. If so, how would it change?

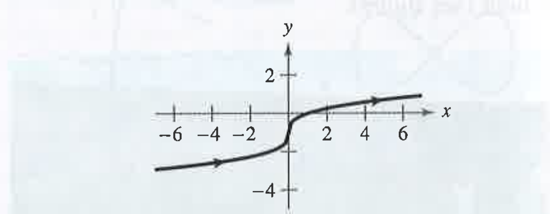


(a) $x = -t,\ y = t^2$

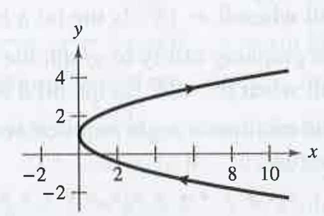
(b) $x = t + 1,\ y = t^2$

(c) $x = t,\ y = t^2 + 1$

109. Think About It The graph of the parametric equations $x = t^3$ and $y = t - 1$ is shown below. Would the graph change for the parametric equations $x = (-t)^3$ and $y = -t - 1$? If so, how would it change?



110. Think About It The graph of the parametric equations $x = t^2$ and $y = t + 1$ is shown below. Would the graph change for the parametric equations $x = (t+1)^2$ and $y = t + 2$? If so, how would it change?



10.7 Polar Coordinates

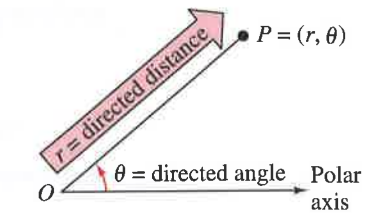
Polar coordinates are often useful tools in mathematical modeling. For example, in Exercise 109 on page 750, you will use polar coordinates to write an equation that models the position of a passenger car on a Ferris wheel.

- Plot points in the polar coordinate system.
- Convert points from rectangular to polar form and vice versa.
- Convert equations from rectangular to polar form and vice versa.

Introduction

So far, you have been representing graphs of equations as collections of points (x, y) in the rectangular coordinate system, where x and y represent the directed distances from the coordinate axes to the point (x, y). In this section, you will study a different system called the **polar coordinate system.**

To form the polar coordinate system in the plane, fix a point O, called the **pole** (or **origin**), and construct from O an initial ray called the **polar axis,** as shown in the figure at the right. Then each point P in the plane can be assigned **polar coordinates** (r, θ), where r and θ are defined below.



1. r = *directed distance* from O to P
2. θ = *directed angle,* counterclockwise from the polar axis to segment $\overline{OP}$

EXAMPLE 1 Plotting Points in the Polar Coordinate System

Plot each point given in polar coordinates.

a. $(2, \pi/3)$ **b.** $(3, -\pi/6)$ **c.** $(3, 11\pi/6)$

Solution

a. The point $(r, \theta) = (2, \pi/3)$ lies two units from the pole on the terminal side of the angle $\theta = \pi/3$, as shown in Figure 10.45.

b. The point $(r, \theta) = (3, -\pi/6)$ lies three units from the pole on the terminal side of the angle $\theta = -\pi/6$, as shown in Figure 10.46.

c. The point $(r, \theta) = (3, 11\pi/6)$ coincides with the point $(3, -\pi/6)$, as shown in Figure 10.47.

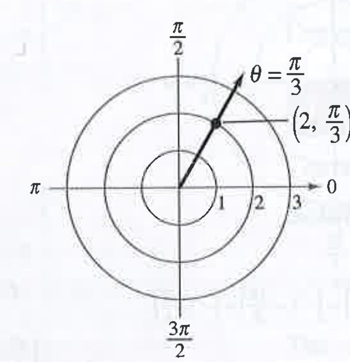


Figure 10.45

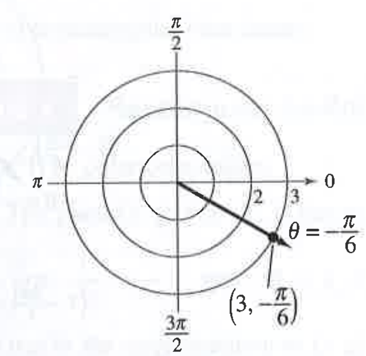


Figure 10.46

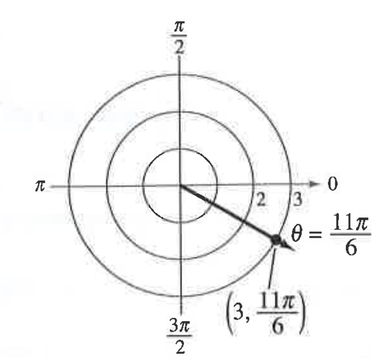


Figure 10.47

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Plot each point given in polar coordinates.

a. $(3, \pi/4)$ **b.** $(2, -\pi/3)$ **c.** $(2, 5\pi/3)$