Application

Kepler's Laws (listed below), named after the German astronomer Johannes Kepler (1571–1630), can be used to describe the orbits of the planets about the sun.

1. Each planet moves in an elliptical orbit with the sun at one focus.
2. A ray from the sun to a planet sweeps out equal areas in equal times.
3. The square of the period (the time it takes for a planet to orbit the sun) is proportional to the cube of the mean distance between the planet and the sun.

Although Kepler stated these laws on the basis of observation, Isaac Newton (1642–1727) later validated them. In fact, Newton showed that these laws apply to the orbits of all heavenly bodies, including comets and satellites. The next example, which involves the comet named after the English mathematician and physicist Edmund Halley (1656–1742), illustrates this.

If you use Earth as a reference with a period of 1 year and a distance of 1 astronomical unit (about 93 million miles), then the proportionality constant in Kepler's third law is 1. For example, Mars has a mean distance to the sun of $d \approx 1.524$ astronomical units. Solve for its period P in $d^3 = P^2$ to find that the period of Mars is $P \approx 1.88$ years.

EXAMPLE 4 **Halley's Comet**

Halley's comet has an elliptical orbit with an eccentricity of $e \approx 0.967$. The length of the major axis of the orbit is approximately 35.88 astronomical units. Find a polar equation for the orbit. How close does Halley's comet come to the sun?

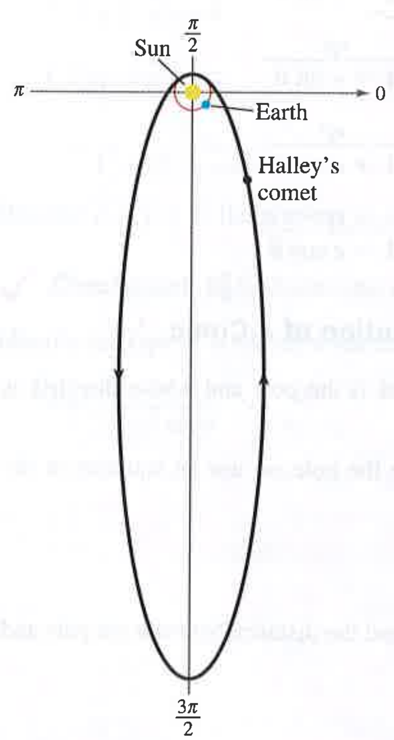


Figure 10.61

Solution Using a vertical major axis, as shown in Figure 10.61, choose an equation of the form $r = ep/(1 + e \sin \theta)$. The vertices of the ellipse occur when $\theta = \pi/2$ and $\theta = 3\pi/2$, and the length of the major axis is the sum of the r-values of the vertices. That is,

$$2a = \frac{0.967p}{1 + 0.967} + \frac{0.967p}{1 - 0.967} \approx 29.79p \approx 35.88.$$

So, $p \approx 1.204$ and $ep \approx (0.967)(1.204) \approx 1.164$. Substituting this value for ep in the equation, you have

$$r = \frac{1.164}{1 + 0.967 \sin \theta}$$

where r is measured in astronomical units. To find the closest point to the sun (a focus), substitute $\theta = \pi/2$ into this equation to obtain

$$r = \frac{1.164}{1 + 0.967 \sin(\pi/2)} \approx 0.59 \text{ astronomical unit} \approx 55{,}000{,}000 \text{ miles.}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Encke's comet has an elliptical orbit with an eccentricity of $e \approx 0.847$. The length of the major axis of the orbit is approximately 4.420 astronomical units. Find a polar equation for the orbit. How close does Encke's comet come to the sun?

Summarize **(Section 10.9)**

1. State the definition of a conic in terms of eccentricity *(page 759)*. For examples of writing and graphing polar equations of conics, see Examples 2 and 3.
2. Describe a real-life application of an equation of a conic in polar form *(page 762, Example 4)*.

10.9 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary

In Exercises 1–3, fill in the blanks.

1. The locus of a point in the plane that moves such that its distance from a fixed point (focus) is in a constant ratio to its distance from a fixed line (directrix) is a ________.
2. The constant ratio is the ________ of the conic and is denoted by ________.
3. An equation of the form $r = \dfrac{ep}{1 - e \cos \theta}$ has a ________ directrix to the ________ of the pole.
4. Match the conic with its eccentricity.
   (a) $0 < e < 1$ (b) $e = 1$ (c) $e > 1$
   (i) Parabola (ii) Hyperbola (iii) Ellipse

Skills and Applications

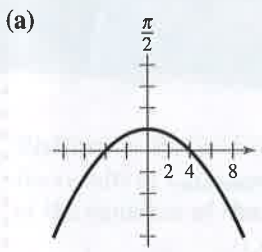
Identifying a Conic **In Exercises 5–8, write the polar equation of the conic for each value of e. Identify the type of conic represented by each equation. Verify your answers with a graphing utility.**

(a) $e = 1$ (b) $e = 0.5$ (c) $e = 1.5$

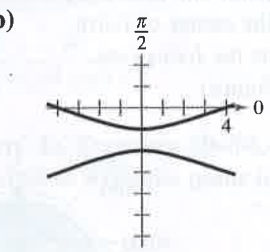
5. $r = \dfrac{2e}{1 + e \cos \theta}$
6. $r = \dfrac{2e}{1 - e \cos \theta}$
7. $r = \dfrac{2e}{1 - e \sin \theta}$
8. $r = \dfrac{2e}{1 + e \sin \theta}$

Matching **In Exercises 9–12, match the polar equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

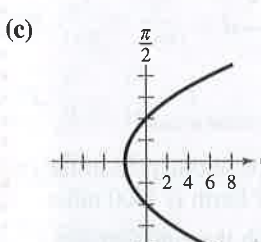
(a)



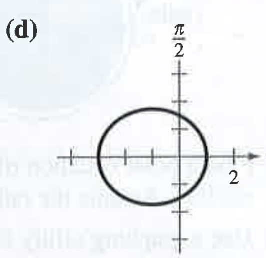
(b)



(c)



(d)



9. $r = \dfrac{4}{1 - \cos \theta}$
10. $r = \dfrac{3}{2 + \cos \theta}$
11. $r = \dfrac{4}{1 + \sin \theta}$
12. $r = \dfrac{4}{1 - 3 \sin \theta}$

Sketching a Conic **In Exercises 13–24, identify the conic represented by the equation and sketch its graph.**

13. $r = \dfrac{3}{1 - \cos \theta}$
14. $r = \dfrac{7}{1 + \sin \theta}$
15. $r = \dfrac{5}{1 - \sin \theta}$
16. $r = \dfrac{6}{1 + \cos \theta}$
17. $r = \dfrac{2}{2 - \cos \theta}$
18. $r = \dfrac{4}{4 + \sin \theta}$
19. $r = \dfrac{6}{2 + \sin \theta}$
20. $r = \dfrac{6}{3 - 2 \sin \theta}$
21. $r = \dfrac{3}{2 + 4 \sin \theta}$
22. $r = \dfrac{5}{-1 + 2 \cos \theta}$
23. $r = \dfrac{3}{2 - 6 \cos \theta}$
24. $r = \dfrac{3}{2 + 6 \sin \theta}$

Graphing a Polar Equation **In Exercises 25–32, use a graphing utility to graph the polar equation. Identify the conic.**

25. $r = \dfrac{-1}{1 - \sin \theta}$
26. $r = \dfrac{-5}{2 + 4 \sin \theta}$
27. $r = \dfrac{3}{-4 + 2 \cos \theta}$
28. $r = \dfrac{4}{1 - 2 \cos \theta}$
29. $r = \dfrac{4}{3 - \cos \theta}$
30. $r = \dfrac{10}{1 + \cos \theta}$
31. $r = \dfrac{14}{14 + 17 \sin \theta}$
32. $r = \dfrac{12}{2 - \cos \theta}$

Graphing a Rotated Conic In Exercises 33–36, use a graphing utility to graph the rotated conic.

33. $r = \dfrac{3}{1 - \cos[\theta - (\pi/4)]}$ (See Exercise 13.)

34. $r = \dfrac{4}{4 + \sin[\theta - (\pi/3)]}$ (See Exercise 18.)

35. $r = \dfrac{6}{2 + \sin[\theta + (\pi/6)]}$ (See Exercise 19.)

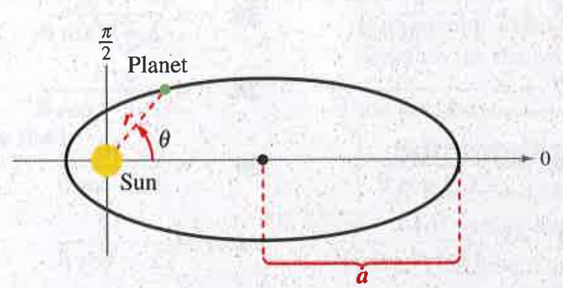
36. $r = \dfrac{3}{2 + 6\sin[\theta + (2\pi/3)]}$ (See Exercise 24.)

Finding the Polar Equation of a Conic In Exercises 37–52, find a polar equation of the indicated conic with the given characteristics and focus at the pole.

	Conic	Eccentricity	Directrix
37.	Parabola	$e = 1$	$x = -1$
38.	Parabola	$e = 1$	$y = -4$
39.	Ellipse	$e = \frac{1}{2}$	$x = 3$
40.	Ellipse	$e = \frac{3}{4}$	$y = -2$
41.	Hyperbola	$e = 2$	$x = 1$
42.	Hyperbola	$e = \frac{3}{2}$	$y = -2$

	Conic	Vertex or Vertices
43.	Parabola	$(2, 0)$
44.	Parabola	$(10, \pi/2)$
45.	Parabola	$(5, \pi)$
46.	Parabola	$(1, -\pi/2)$
47.	Ellipse	$(2, 0), (10, \pi)$
48.	Ellipse	$(2, \pi/2), (4, 3\pi/2)$
49.	Ellipse	$(20, 0), (4, \pi)$
50.	Hyperbola	$(2, 0), (8, 0)$
51.	Hyperbola	$(1, 3\pi/2), (9, 3\pi/2)$
52.	Hyperbola	$(4, \pi/2), (1, \pi/2)$

53. Astronomy The planets travel in elliptical orbits with the sun at one focus. Assume that the focus is at the pole, the major axis lies on the polar axis, and the length of the major axis is $2a$ (see figure). Show that the polar equation of the orbit is $r = a(1 - e^2)/(1 - e\cos\theta)$, where e is the eccentricity.



54. Astronomy Use the result of Exercise 53 to show that the minimum distance *(perihelion)* from the sun to the planet is

$$r = a(1 - e)$$

and the maximum distance *(aphelion)* is

$$r = a(1 + e).$$

Planetary Motion In Exercises 55–60, use the results of Exercises 53 and 54 to find (a) the polar equation of the planet's orbit and (b) the perihelion and aphelion.

55. Earth $a \approx 9.2957 \times 10^7$ miles, $e \approx 0.0167$

56. Saturn $a \approx 1.4335 \times 10^9$ kilometers, $e \approx 0.0565$

57. Venus $a \approx 1.0821 \times 10^8$ kilometers, $e \approx 0.0067$

58. Mercury $a \approx 3.5984 \times 10^7$ miles, $e \approx 0.2056$

59. Mars $a \approx 1.4162 \times 10^8$ miles, $e \approx 0.0935$

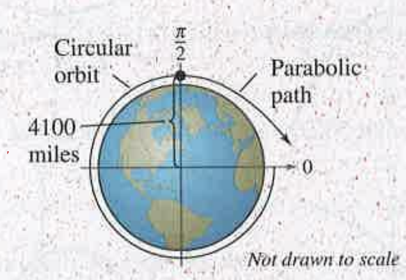
60. Jupiter $a \approx 7.7857 \times 10^8$ kilometers, $e \approx 0.0489$

61. Error Analysis Describe the error.

For the polar equation $r = \dfrac{3}{2 + \sin\theta}$, $e = 1$.
So, the equation represents a parabola.

62. Satellite Orbit

A satellite in a 100-mile-high circular orbit around Earth has a velocity of approximately 17,500 miles per hour. If this velocity is multiplied by $\sqrt{2}$, then the satellite will have the minimum velocity necessary to escape Earth's gravity and will follow a parabolic path with the center of Earth as the focus (see figure).



(a) Find a polar equation of the parabolic path of the satellite. Assume the radius of Earth is 4000 miles.

(b) Use a graphing utility to graph the equation you found in part (a).

(c) Find the distance between the surface of the Earth and the satellite when $\theta = 30°$.

(d) Find the distance between the surface of Earth and the satellite when $\theta = 60°$.

Exploration

True or False? In Exercises 63–66, determine whether the statement is true or false. Justify your answer.

63. For values of $e > 1$ and $0 \le \theta \le 2\pi$, the graphs of

$$r = \frac{ex}{1 - e\cos\theta} \quad \text{and} \quad r = \frac{e(-x)}{1 + e\cos\theta}$$

are the same.

64. The graph of

$$r = \frac{4}{-3 - 3\sin\theta}$$

has a horizontal directrix above the pole.

65. The conic represented by

$$r^2 = \frac{16}{9 - 4\cos\left(\theta + \frac{\pi}{4}\right)}$$

is an ellipse.

66. The conic represented by

$$r = \frac{6}{3 - 2\cos\theta}$$

is a parabola.

67. Verifying a Polar Equation Show that the polar equation of the ellipse represented by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{is} \quad r^2 = \frac{b^2}{1 - e^2\cos^2\theta}.$$

68. Verifying a Polar Equation Show that the polar equation of the hyperbola represented by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{is} \quad r^2 = \frac{-b^2}{1 - e^2\cos^2\theta}.$$

Writing a Polar Equation In Exercises 69–74, use the results of Exercises 67 and 68 to write the polar form of the equation of the conic.

69. $\dfrac{x^2}{169} + \dfrac{y^2}{144} = 1$ **70.** $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

71. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

72. $\dfrac{x^2}{36} - \dfrac{y^2}{4} = 1$

73. Hyperbola
One focus: $(5, 0)$
Vertices: $(4, 0), (4, \pi)$

74. Ellipse
One focus: $(4, 0)$
Vertices: $(5, 0), (5, \pi)$

75. Writing Explain how the graph of each equation differs from the conic represented by $r = \dfrac{5}{1 - \sin\theta}$. (See Exercise 15.)

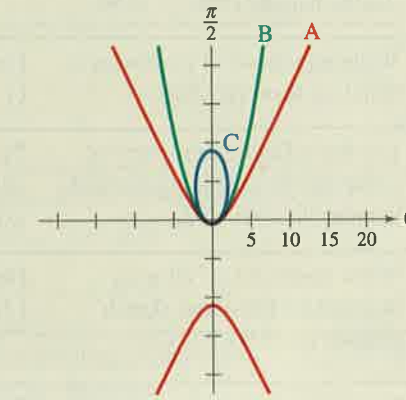
(a) $r = \dfrac{5}{1 - \cos\theta}$ (b) $r = \dfrac{5}{1 + \sin\theta}$

(c) $r = \dfrac{5}{1 + \cos\theta}$ (d) $r = \dfrac{5}{1 - \sin[\theta - (\pi/4)]}$

76. HOW DO YOU SEE IT? The graph of

$$r = \frac{e}{1 - e\sin\theta}$$

is shown for different values of e. Determine which graph matches each value of e.



(a) $e = 0.9$ (b) $e = 1.0$ (c) $e = 1.1$

77. Reasoning

(a) Identify the type of conic represented by

$$r = \frac{4}{1 - 0.4\cos\theta}$$

without graphing the equation.

(b) Without graphing the equations, describe how the graph of each equation below differs from the polar equation given in part (a).

$$r_1 = \frac{4}{1 + 0.4\cos\theta} \qquad r_2 = \frac{4}{1 - 0.4\sin\theta}$$

(c) Use a graphing utility to verify your results in part (b).

78. Reasoning The equation

$$r = \frac{ep}{1 \pm e\sin\theta}$$

represents an ellipse with $e < 1$. What happens to the lengths of both the major axis and the minor axis when the value of e remains fixed and the value of p changes? Use an example to explain your reasoning.