EXAMPLE 9 Finding a Minimum Area

A rectangular page contains 48 square inches of print. The margins at the top and bottom of the page are each 1 inch deep. The margins on each side are $1\frac{1}{2}$ inches wide. What should the dimensions of the page be to use the least amount of paper?

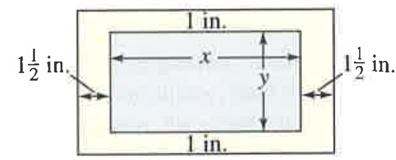


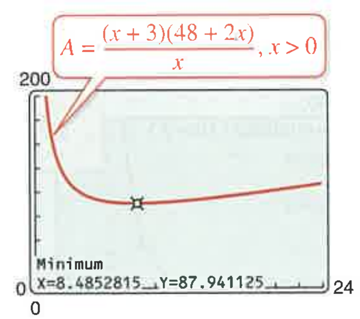
Figure 2.28

Graphical Solution

Let A be the area to be minimized. From Figure 2.28, you can write $A = (x + 3)(y + 2)$. The printed area inside the margins is given by $xy = 48$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

The graph of this rational function is shown below. Because x represents the width of the printed area, you need to consider only the portion of the graph for which x is positive. Use the *minimum* feature of a graphing utility to estimate that the minimum value of A occurs when $x \approx 8.5$ inches. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be $x + 3 \approx 11.5$ inches by $y + 2 \approx 7.6$ inches.



Numerical Solution

Let A be the area to be minimized. From Figure 2.28, you can write $A = (x + 3)(y + 2)$. The printed area inside the margins is given by $xy = 48$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

Use the *table* feature of a graphing utility to create a table of values for the function $y_1 = [(x + 3)(48 + 2x)]/x$ beginning at $x = 1$ and increasing by 1. The minimum value of y_1 occurs when x is somewhere between 8 and 9, as shown in Figure 2.29. To approximate the minimum value of y_1 to one decimal place, change the table to begin at $x = 8$ and increase by 0.1. The minimum value of y_1 occurs when $x \approx 8.5$, as shown in Figure 2.30. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be $x + 3 \approx 11.5$ inches by $y + 2 \approx 7.6$ inches.

X	Y1
6	90
7	88.571
8	88
9	88
10	88.4
11	89.091
12	90
X=8	

Figure 2.29

X	Y1
8.2	87.961
8.3	87.949
8.4	87.943
8.5	87.941
8.6	87.944
8.7	87.952
8.8	87.964
X=8.5	

Figure 2.30

✓ ***Checkpoint*** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Rework Example 9 when the margins on each side are 2 inches wide and the page contains 40 square inches of print.

Summarize (Section 2.6)

1. State the definition of a rational function and describe the domain *(page 166)*. For an example of finding the domain of a rational function, see Example 1.
2. Explain how to find the vertical and horizontal asymptotes of the graph of a rational function *(page 168)*. For an example of finding vertical and horizontal asymptotes of graphs of rational functions, see Example 2.
3. Explain how to sketch the graph of a rational function *(page 169)*. For examples of sketching the graphs of rational functions, see Examples 3–6.
4. Explain how to determine whether the graph of a rational function has a slant asymptote *(page 172)*. For an example of sketching the graph of a rational function that has a slant asymptote, see Example 7.
5. Describe examples of how to use rational functions to model and solve real-life problems *(pages 173 and 174, Examples 8 and 9)*.

2.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. Functions of the form $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial, are called ________ ________.
2. When $f(x) \to \pm\infty$ as $x \to a$ from the left or the right, $x = a$ is a ________ ________ of the graph of f.
3. When $f(x) \to b$ as $x \to \pm\infty$, $y = b$ is a ________ ________ of the graph of f.
4. For the rational function $f(x) = N(x)/D(x)$, if the degree of $N(x)$ is exactly one more than the degree of $D(x)$, then the graph of f has a ________ (or oblique) ________.

Skills and Applications

Finding the Domain of a Rational Function In Exercises 5–8, find the domain of the function and discuss the behavior of f near any excluded x-values.

5. $f(x) = \dfrac{1}{x - 1}$
6. $f(x) = \dfrac{5x}{x + 2}$
7. $f(x) = \dfrac{3x^2}{x^2 - 1}$
8. $f(x) = \dfrac{2x}{x^2 - 4}$

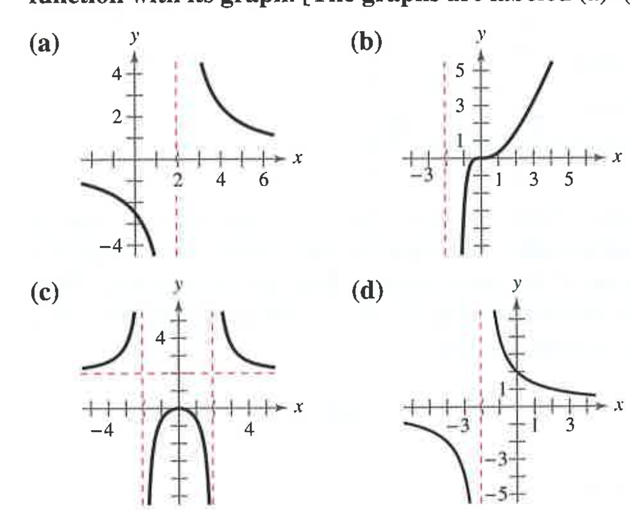
Finding Vertical and Horizontal Asymptotes In Exercises 9–16, find all vertical and horizontal asymptotes of the graph of the function.

9. $f(x) = \dfrac{4}{x^2}$
10. $f(x) = \dfrac{1}{(x - 2)^3}$
11. $f(x) = \dfrac{5 + x}{5 - x}$
12. $f(x) = \dfrac{3 - 7x}{3 + 2x}$
13. $f(x) = \dfrac{x^3}{x^2 - x}$
14. $f(x) = \dfrac{4x^2}{x + 2}$
15. $f(x) = \dfrac{x^2 - 3x - 4}{2x^2 + x - 1}$
16. $f(x) = \dfrac{-4x^2 + 1}{x^2 + x + 3}$

Sketching the Graph of a Rational Function In Exercises 17–38, (a) state the domain of the function, (b) identify all intercepts, (c) find any vertical or horizontal asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.

17. $f(x) = \dfrac{1}{x + 1}$
18. $f(x) = \dfrac{1}{x - 3}$
19. $h(x) = \dfrac{-1}{x + 4}$
20. $g(x) = \dfrac{1}{6 - x}$
21. $C(x) = \dfrac{2x + 3}{x + 2}$
22. $P(x) = \dfrac{1 - 3x}{1 - x}$
23. $f(x) = \dfrac{x^2}{x^2 + 9}$
24. $f(t) = \dfrac{1 - 2t}{t}$
25. $g(s) = \dfrac{4s}{s^2 + 4}$
26. $f(x) = -\dfrac{x}{(x - 2)^2}$
27. $h(x) = \dfrac{2x}{x^2 - 3x - 4}$
28. $g(x) = \dfrac{3x}{x^2 + 2x - 3}$
29. $f(x) = \dfrac{x - 4}{x^2 - 16}$
30. $f(x) = \dfrac{x + 1}{x^2 - 1}$
31. $f(t) = \dfrac{t^2 - 1}{t - 1}$
32. $f(x) = \dfrac{x^2 - 36}{x + 6}$
33. $f(x) = \dfrac{x^2 - 25}{x^2 - 4x - 5}$
34. $f(x) = \dfrac{x^2 - 4}{x^2 - 3x + 2}$
35. $f(x) = \dfrac{x^2 + 3x}{x^2 + x - 6}$
36. $f(x) = \dfrac{5(x + 4)}{x^2 + x - 12}$
37. $f(x) = \dfrac{2x^2 - 5x - 3}{x^3 - 2x^2 - x + 2}$
38. $f(x) = \dfrac{x^2 - x - 2}{x^3 - 2x^2 - 5x + 6}$

Matching In Exercises 39–42, match the rational function with its graph. [The graphs are labeled (a)–(d).]



39. $f(x) = \dfrac{4}{x + 2}$
40. $f(x) = \dfrac{5}{x - 2}$
41. $f(x) = \dfrac{2x^2}{x^2 - 4}$
42. $f(x) = \dfrac{3x^3}{(x + 2)^2}$

Comparing Graphs of Functions **In Exercises 43–46, (a) state the domains of f and g, (b) use a graphing utility to graph f and g in the same viewing window, and (c) explain why the graphing utility may not show the difference in the domains of f and g.**

43. $f(x) = \dfrac{x^2 - 1}{x + 1}$, $g(x) = x - 1$

44. $f(x) = \dfrac{x^2(x - 2)}{x^2 - 2x}$, $g(x) = x$

45. $f(x) = \dfrac{x - 2}{x^2 - 2x}$, $g(x) = \dfrac{1}{x}$

46. $f(x) = \dfrac{2x - 6}{x^2 - 7x + 12}$, $g(x) = \dfrac{2}{x - 4}$

A Rational Function with a Slant Asymptote **In Exercises 47–60, (a) state the domain of the function, (b) identify all intercepts, (c) find any vertical or slant asymptotes, and (d) plot additional solution points as needed to sketch the graph of the rational function.**

47. $h(x) = \dfrac{x^2 - 4}{x}$

48. $g(x) = \dfrac{x^2 + 5}{x}$

49. $f(x) = \dfrac{2x^2 + 1}{x}$

50. $f(x) = \dfrac{-x^2 - 2}{x}$

51. $g(x) = \dfrac{x^2 + 1}{x}$

52. $h(x) = \dfrac{x^2}{x - 1}$

53. $f(t) = -\dfrac{t^2 + 1}{t + 5}$

54. $f(x) = \dfrac{x^2 + 1}{x + 1}$

55. $f(x) = \dfrac{x^3}{x^2 - 4}$

56. $g(x) = \dfrac{x^3}{2x^2 - 8}$

57. $f(x) = \dfrac{x^2 - x + 1}{x - 1}$

58. $f(x) = \dfrac{2x^2 - 5x + 5}{x - 2}$

59. $f(x) = \dfrac{2x^3 - x^2 - 2x + 1}{x^2 + 3x + 2}$

60. $f(x) = \dfrac{2x^3 + x^2 - 8x - 4}{x^2 - 3x + 2}$

Using Technology **In Exercises 61–64, use a graphing utility to graph the rational function. State the domain of the function and find any asymptotes. Then zoom out sufficiently far so that the graph appears as a line. Identify the line.**

61. $f(x) = \dfrac{x^2 + 2x - 8}{x + 2}$

62. $f(x) = \dfrac{2x^2 + x}{x + 1}$

63. $g(x) = \dfrac{1 + 3x^2 - x^3}{x^2}$

64. $h(x) = \dfrac{12 - 2x - x^2}{2(4 + x)}$

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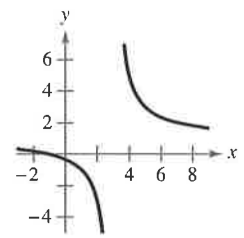
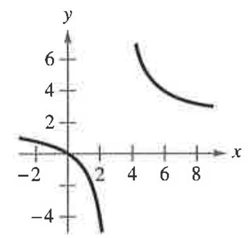
Graphical Reasoning **In Exercises 65–68, (a) use the graph to determine any x-intercepts of the graph of the rational function and (b) set $y = 0$ and solve the resulting equation to confirm your result in part (a).**

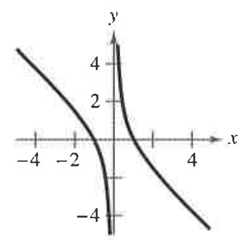
65. $y = \dfrac{x + 1}{x - 3}$



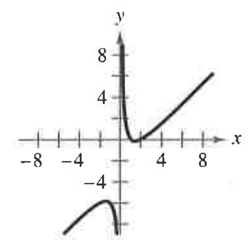
66. $y = \dfrac{2x}{x - 3}$



67. $y = \dfrac{1}{x} - x$



68. $y = x - 3 + \dfrac{2}{x}$



69. **Recycling** The cost C (in dollars) of supplying recycling bins to $p\%$ of the population of a rural township is given by

$$C = \frac{25{,}000p}{100 - p}, \quad 0 \le p < 100.$$

(a) Use a graphing utility to graph the cost function.

(b) Find the costs of supplying bins to 15%, 50%, and 90% of the population.

(c) According to the model, is it possible to supply bins to 100% of the population? Explain.

70. **Population Growth** The game commission introduces 100 deer into newly acquired state game lands. The population N of the herd is modeled by

$$N = \frac{20(5 + 3t)}{1 + 0.04t}, \quad t \ge 0$$

where t is the time in years.

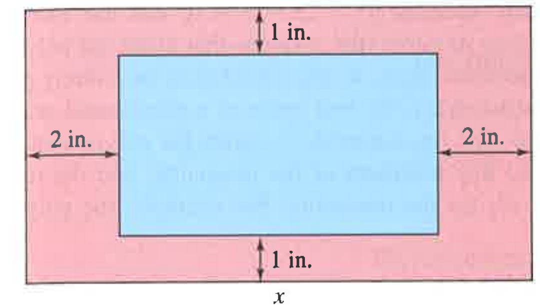
(a) Use a graphing utility to graph this model.

(b) Find the populations when $t = 5$, $t = 10$, and $t = 25$.

(c) What is the limiting size of the herd as time increases?

71. **Page Design** A rectangular page contains 64 square inches of print. The margins at the top and bottom of the page are each 1 inch deep. The margins on each side are $1\frac{1}{2}$ inches wide. What should the dimensions of the page be to use the least amount of paper?

72. **Page Design** A page that is x inches wide and y inches high contains 30 square inches of print. The top and bottom margins are each 1 inch deep, and the margins on each side are 2 inches wide (see figure).



(a) Write a functions for the total area A of the page in terms of x.

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the area function and approximate the dimensions of the page that use the least amount of paper.

73. **Average Speed** A driver's average speed is 50 miles per hour on a round trip between two cities 100 miles apart. The average speeds for going and returning were x and y miles per hour, respectively.

(a) Show that $y = (25x)/(x - 25)$.

(b) Determine the vertical and horizontal asymptotes of the graph of the function.

(c) Use a graphing utility to graph the function.

(d) Complete the table.

x	30	35	40	45	50	55	60
y							

(e) Are the results in the table what you expected? Explain.

(f) Is it possible to average 20 miles per hour in one direction and still average 50 miles per hour on the round trip? Explain.

74. **Medicine** The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by

$$C = \frac{3t^2 + t}{t^3 + 50}, \quad t > 0.$$

Use a graphing utility to graph the function. Determine the horizontal asymptote of the graph of the function and interpret its meaning in the context of the problem.

Exploration

True or False? **In Exercises 75–77, determine whether the statement is true or false. Justify your answer.**

75. The graph of a polynomial function can have infinitely many vertical asymptotes.

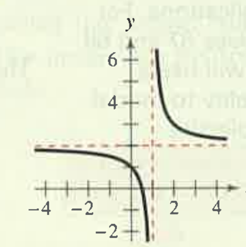
76. The graph of a rational function can never cross one of its asymptotes.

77. The graph of a rational function can have a vertical asymptote, a horizontal asymptote, and a slant asymptote.

78. **HOW DO YOU SEE IT?** The graph of a rational function

$$f(x) = \frac{N(x)}{D(x)}$$

is shown below. Determine which of the statements about the function is false. Justify your answer.



(a) $D(1) = 0$.

(b) The degree of $N(x)$ and $D(x)$ are equal.

(c) The ratio of the leading coefficients of $N(x)$ and $D(x)$ is 1.

79. **Writing** Is every rational function a polynomial function? Is every polynomial function a rational function? Explain.

Writing a Rational Function **In Exercises 80–82, write a rational function f whose graph has the specified characteristics. (There are many correct answers.)**

80. Vertical asymptote: None
Horizontal asymptote: $y = 2$

81. Vertical asymptotes: $x = -2$, $x = 1$
Horizontal asymptote: None

82. Vertical asymptote: $x = 2$
Slant asymptote: $y = x + 1$
Zero of the function: $x = -2$

Project: Department of Defense To work an extended application analyzing the total numbers of military personnel on active duty from 1984 through 2014, visit this text's website at *LarsonPrecalculus.com*. *(Source: U.S. Department of Defense)*