Application

One way to determine a possible relationship between the x- and y-values of a set of nonlinear data is to take the natural logarithm of each x-value and each y-value. If the plotted points $(\ln x, \ln y)$ lie on a line, then x and y are related by the equation $\ln y = m \ln x$, where m is the slope of the line.

EXAMPLE 7 Finding a Mathematical Model

The table shows the mean distance x from the sun and the period y (the time it takes a planet to orbit the sun, in years) for each of the six planets that are closest to the sun. In the table, the mean distance is given in astronomical units (where one astronomical unit is defined as Earth's mean distance from the sun). The points from the table are plotted in Figure 3.13. Find an equation that relates y and x.

DATA

Planet	Mean Distance, x	Period, y
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.524	1.881
Jupiter	5.203	11.862
Saturn	9.537	29.457

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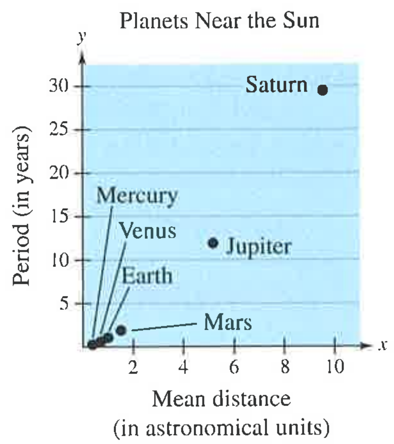

Figure 3.13

Solution From Figure 3.13, it is not clear how to find an equation that relates y and x. To solve this problem, make a table of values giving the natural logarithms of all x- and y-values of the data (see the table at the left). Plot each point $(\ln x, \ln y)$. These points appear to lie on a line (see Figure 3.14). Choose two points to determine the slope of the line. Using the points $(0.421, 0.632)$ and $(0, 0)$, the slope of the line is

$$m = \frac{0.632 - 0}{0.421 - 0} \approx 1.5 = \frac{3}{2}.$$

By the point-slope form, the equation of the line is $Y = \frac{3}{2}X$, where $Y = \ln y$ and $X = \ln x$. So, an equation that relates y and x is $\ln y = \frac{3}{2} \ln x$.

Planet	$\ln x$	$\ln y$
Mercury	-0.949	-1.423
Venus	-0.324	-0.486
Earth	0.000	0.000
Mars	0.421	0.632
Jupiter	1.649	2.473
Saturn	2.255	3.383

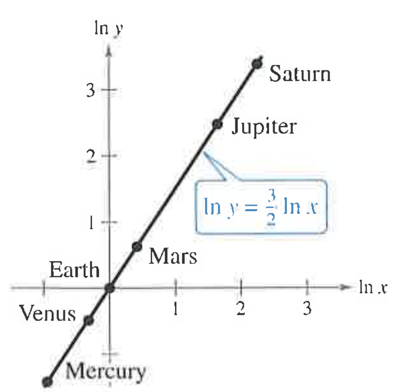

Figure 3.14

✓ ***Checkpoint*** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Find a logarithmic equation that relates y and x for the following ordered pairs.

$(0.37, 0.51)$, $(1.00, 1.00)$, $(2.72, 1.95)$, $(7.39, 3.79)$, $(20.09, 7.39)$

Summarize (Section 3.3)

1. State the change-of-base formula *(page 219)*. For examples of using the change-of-base formula to rewrite and evaluate logarithmic expressions, see Examples 1 and 2.
2. Make a list of the properties of logarithms *(page 220)*. For examples of using the properties of logarithms to evaluate or rewrite logarithmic expressions, see Examples 3 and 4.
3. Explain how to use the properties of logarithms to expand or condense logarithmic expressions *(page 221)*. For examples of expanding and condensing logarithmic expressions, see Examples 5 and 6.
4. Describe an example of how to use a logarithmic function to model and solve a real-life problem *(page 222, Example 7)*.

3.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary

In Exercises 1–3, fill in the blanks.

1. To evaluate a logarithm to any base, use the ________ formula.
2. The change-of-base formula for base e is $\log_a x =$ ________.
3. When you consider $\log_a x$ to be a constant multiple of $\log_b x$, the constant multiplier is ________.
4. Name the property of logarithms illustrated by each statement.

(a) $\ln(uv) = \ln u + \ln v$ (b) $\log_a u^n = n \log_a u$ (c) $\ln \frac{u}{v} = \ln u - \ln v$

Skills and Applications

Changing Bases **In Exercises 5–8, rewrite the logarithm as a ratio of (a) common logarithms and (b) natural logarithms.**

5. $\log_5 16$
6. $\log_{1/5} 4$
7. $\log_x \frac{3}{10}$
8. $\log_{2.6} x$

Using the Change-of-Base Formula **In Exercises 9–12, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.**

9. $\log_3 17$
10. $\log_{0.4} 12$
11. $\log_\pi 0.5$
12. $\log_{2/3} 0.125$

Using Properties of Logarithms **In Exercises 13–18, use the properties of logarithms to write the logarithm in terms of $\log_3 5$ and $\log_3 7$.**

13. $\log_3 35$
14. $\log_3 \frac{5}{7}$
15. $\log_3 \frac{7}{25}$
16. $\log_3 175$
17. $\log_3 \frac{21}{5}$
18. $\log_3 \frac{45}{49}$

Using Properties of Logarithms **In Exercises 19–32, find the exact value of the logarithmic expression without using a calculator. (If this is not possible, state the reason.)**

19. $\log_3 9$
20. $\log_5 \frac{1}{125}$
21. $\log_6 \sqrt[3]{\frac{1}{6}}$
22. $\log_2 \sqrt[4]{8}$
23. $\log_2(-2)$
24. $\log_3(-27)$
25. $\ln \sqrt[4]{e^3}$
26. $\ln(1/\sqrt{e})$
27. $\ln e^2 + \ln e^5$
28. $2 \ln e^6 - \ln e^5$
29. $\log_5 75 - \log_5 3$
30. $\log_4 2 + \log_4 32$
31. $\log_4 8$
32. $\log_8 16$

Using Properties of Logarithms **In Exercises 33–40, approximate the logarithm using the properties of logarithms, given $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$.**

33. $\log_b 10$
34. $\log_b \frac{2}{3}$
35. $\log_b 0.04$
36. $\log_b \sqrt{2}$
37. $\log_b 45$
38. $\log_b(3b^2)$
39. $\log_b(2b)^{-2}$
40. $\log_b \sqrt[3]{3b}$

Expanding a Logarithmic Expression **In Exercises 41–60, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)**

41. $\ln 7x$
42. $\log_3 13z$
43. $\log_8 x^4$
44. $\ln(xy)^3$
45. $\log_5 \frac{5}{x}$
46. $\log_6 \frac{w^2}{v}$
47. $\ln \sqrt{z}$
48. $\ln \sqrt[3]{t}$
49. $\ln xyz^2$
50. $\log_4 11b^2c$
51. $\ln z(z-1)^2, \quad z > 1$
52. $\ln \frac{x^2 - 1}{x^3}, \quad x > 1$
53. $\log_2 \frac{\sqrt{a^2 - 4}}{7}, \quad a > 2$
54. $\ln \frac{3}{\sqrt{x^2 + 1}}$
55. $\log_5 \frac{x^2}{y^2 z^3}$
56. $\log_{10} \frac{xy^4}{z^5}$
57. $\ln \sqrt[3]{\frac{yz}{x^2}}$
58. $\log_2 x^4 \sqrt{\frac{y}{z^3}}$
59. $\ln \sqrt[4]{x^3(x^2 + 3)}$
60. $\ln \sqrt{x^2(x + 2)}$

Condensing a Logarithmic Expression **In Exercises 61–76, condense the expression to the logarithm of a single quantity.**

61. $\ln 3 + \ln x$

62. $\log_5 8 - \log_5 t$

63. $\frac{2}{3}\log_7(z-2)$

64. $-4 \ln 3x$

65. $\log_3 5x - 4\log_3 x$

66. $2\log_2 x + 4\log_2 y$

67. $\log x + 2\log(x+1)$

68. $2\ln 8 - 5\ln(z-4)$

69. $\log x - 2\log y + 3\log z$

70. $3\log_3 x + \frac{1}{4}\log_3 y - 4\log_3 z$

71. $\ln x - [\ln(x+1) + \ln(x-1)]$

72. $4[\ln z + \ln(z+5)] - 2\ln(z-5)$

73. $\frac{1}{2}[2\ln(x+3) + \ln x - \ln(x^2-1)]$

74. $2[3\ln x - \ln(x+1) - \ln(x-1)]$

75. $\frac{1}{3}[\log_8 y + 2\log_8(y+4)] - \log_8(y-1)$

76. $\frac{1}{2}[\log_4(x+1) + 2\log_4(x-1)] + 6\log_4 x$

Comparing Logarithmic Quantities **In Exercises 77 and 78, determine which (if any) of the logarithmic expressions are equal. Justify your answer.**

77. $\dfrac{\log_2 32}{\log_2 4}$, $\log_2 \dfrac{32}{4}$, $\log_2 32 - \log_2 4$

78. $\log_7 \sqrt{70}$, $\log_7 35$, $\frac{1}{2} + \log_7 \sqrt{10}$

Sound Intensity

In Exercises 79–82, use the following information. The relationship between the number of decibels β and the intensity of a sound I (in watts per square meter) is

$$\beta = 10 \log \frac{I}{10^{-12}}.$$

79. Use the properties of logarithms to write the formula in a simpler form. Then determine the number of decibels of a sound with an intensity of 10^{-6} watt per square meter.

80. Find the difference in loudness between an average office with an intensity of 1.26×10^{-7} watt per square meter and a broadcast studio with an intensity of 3.16×10^{-10} watt per square meter.

81. Find the difference in loudness between a vacuum cleaner with an intensity of 10^{-4} watt per square meter and rustling leaves with an intensity of 10^{-11} watt per square meter.

82. You and your roommate are playing your stereos at the same time and at the same intensity. How much louder is the music when both stereos are playing compared with just one stereo playing?

Curve Fitting **In Exercises 83–86, find a logarithmic equation that relates y and x.**

83.

x	1	2	3	4	5	6
y	1	1.189	1.316	1.414	1.495	1.565

84.

x	1	2	3	4	5	6
y	1	0.630	0.481	0.397	0.342	0.303

85.

x	1	2	3	4	5	6
y	2.5	2.102	1.9	1.768	1.672	1.597

86.

x	1	2	3	4	5	6
y	0.5	2.828	7.794	16	27.951	44.091

87. Stride Frequency of Animals Four-legged animals run with two different types of motion: trotting and galloping. An animal that is trotting has at least one foot on the ground at all times, whereas an animal that is galloping has all four feet off the ground at some point in its stride. The number of strides per minute at which an animal breaks from a trot to a gallop depends on the weight of the animal. Use the table to find a logarithmic equation that relates an animal's weight x (in pounds) and its lowest stride frequency while galloping y (in strides per minute).

DATA

Weight, x	Stride Frequency, y
25	191.5
35	182.7
50	173.8
75	164.2
500	125.9
1000	114.2

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88. Nail Length The approximate lengths and diameters (in inches) of bright common wire nails are shown in the table. Find a logarithmic equation that relates the diameter y of a bright common wire nail to its length x.

Length, x	Diameter, y
2	0.113
3	0.148
4	0.192
5	0.225
6	0.262

89. Comparing Models A cup of water at an initial temperature of 78°C is placed in a room at a constant temperature of 21°C. The temperature of the water is measured every 5 minutes during a half-hour period. The results are recorded as ordered pairs of the form (t, T), where t is the time (in minutes) and T is the temperature (in degrees Celsius).

$(0, 78.0°)$, $(5, 66.0°)$, $(10, 57.5°)$, $(15, 51.2°)$, $(20, 46.3°)$, $(25, 42.4°)$, $(30, 39.6°)$

(a) Subtract the room temperature from each of the temperatures in the ordered pairs. Use a graphing utility to plot the data points (t, T) and $(t, T-21)$.

(b) An exponential model for the data $(t, T-21)$ is $T - 21 = 54.4(0.964)^t$. Solve for T and graph the model. Compare the result with the plot of the original data.

(c) Use the graphing utility to plot the points $(t, \ln(T-21))$ and observe that the points appear to be linear. Use the *regression* feature of the graphing utility to fit a line to these data. This resulting line has the form $\ln(T-21) = at + b$, which is equivalent to $e^{\ln(T-21)} = e^{at+b}$. Solve for T, and verify that the result is equivalent to the model in part (b).

(d) Fit a rational model to the data. Take the reciprocals of the y-coordinates of the revised data points to generate the points

$$\left(t, \frac{1}{T-21}\right).$$

Use the graphing utility to graph these points and observe that they appear to be linear. Use the *regression* feature of the graphing utility to fit a line to these data. The resulting line has the form

$$\frac{1}{T-21} = at + b.$$

Solve for T, and use the graphing utility to graph the rational function and the original data points.

90. Writing Write a short paragraph explaining why the transformations of the data in Exercise 89 were necessary to obtain the models. Why did taking the logarithms of the temperatures lead to a linear scatter plot? Why did taking the reciprocals of the temperatures lead to a linear scatter plot?

Exploration

True or False? **In Exercises 91–96, determine whether the statement is true or false given that $f(x) = \ln x$. Justify your answer.**

91. $f(0) = 0$

92. $f(ax) = f(a) + f(x), \quad a > 0, \quad x > 0$

93. $f(x-2) = f(x) - f(2), \quad x > 2$

94. $\sqrt{f(x)} = \frac{1}{2}f(x)$

95. If $f(u) = 2f(v)$, then $v = u^2$.

96. If $f(x) < 0$, then $0 < x < 1$.

Using the Change-of-Base Formula **In Exercises 97–100, use the change-of-base formula to rewrite the logarithm as a ratio of logarithms. Then use a graphing utility to graph the ratio.**

97. $f(x) = \log_2 x$

98. $f(x) = \log_{1/2} x$

99. $f(x) = \log_{1/4} x$

100. $f(x) = \log_{11.8} x$

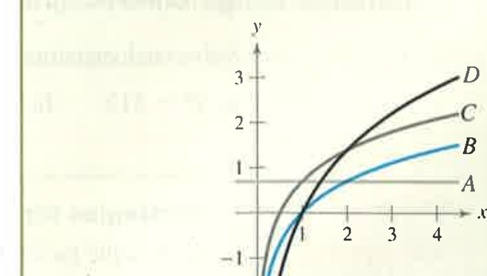
Error Analysis **In Exercises 101 and 102, describe the error.**

101. $(\ln e)^2 = 2(\ln e) = 2(1) = 2$

102.
$$\begin{aligned}\log_2 8 &= \log_2(4+4)\\ &= \log_2 4 + \log_2 4\\ &= \log_2 2^2 + \log_2 2^2\\ &= 2+2\\ &= 4\end{aligned}$$

103. Graphical Reasoning Use a graphing utility to graph the functions $y_1 = \ln x - \ln(x-3)$ and $y_2 = \ln \dfrac{x}{x-3}$ in the same viewing window. Does the graphing utility show the functions with the same domain? If not, explain why some numbers are in the domain of one function but not the other.

104. HOW DO YOU SEE IT? The figure shows the graphs of $y = \ln x$, $y = \ln x^2$, $y = \ln 2x$, and $y = \ln 2$. Match each function with its graph. (The graphs are labeled A through D.) Explain.



105. Think About It For which integers between 1 and 20 can you approximate natural logarithms, given the values $\ln 2 \approx 0.6931$, $\ln 3 \approx 1.0986$, and $\ln 5 \approx 1.6094$? Approximate these logarithms. (Do not use a calculator.)