EXAMPLE 8 Trigonometric Substitution

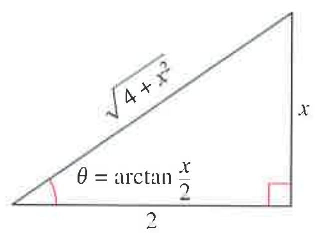
Use the substitution $x = 2\tan\theta$, $0 < \theta < \pi/2$, to write $\sqrt{4 + x^2}$ as a trigonometric function of θ.

Solution Begin by letting $x = 2\tan\theta$. Then, you obtain

$$\begin{aligned}
\sqrt{4+x^2} &= \sqrt{4 + (2\tan\theta)^2} && \text{Substitute } 2\tan\theta \text{ for } x.\\
&= \sqrt{4 + 4\tan^2\theta} && \text{Property of exponents}\\
&= \sqrt{4(1+\tan^2\theta)} && \text{Factor.}\\
&= \sqrt{4\sec^2\theta} && \text{Pythagorean identity}\\
&= 2\sec\theta. && \sec\theta > 0 \text{ for } 0 < \theta < \tfrac{\pi}{2}
\end{aligned}$$

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Use the substitution $x = 3\sin\theta$, $0 < \theta < \pi/2$, to write $\sqrt{9 - x^2}$ as a trigonometric function of θ.



$2\tan\theta = x \Rightarrow \tan\theta = \frac{x}{2}$

Figure 5.1

Figure 5.1 shows the right triangle illustration of the trigonometric substitution $x = 2\tan\theta$ in Example 8. You can use this triangle to check the solution to Example 8. For $0 < \theta < \pi/2$, you have

$$\text{opp} = x, \quad \text{adj} = 2, \quad \text{and} \quad \text{hyp} = \sqrt{4+x^2}.$$

Using these expressions,

$$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{4+x^2}}{2}.$$

So, $2\sec\theta = \sqrt{4+x^2}$, and the solution checks.

EXAMPLE 9 Rewriting a Logarithmic Expression

Rewrite $\ln|\csc\theta| + \ln|\tan\theta|$ as a single logarithm and simplify the result.

Solution

$$\begin{aligned}
\ln|\csc\theta| + \ln|\tan\theta| &= \ln|\csc\theta\tan\theta| && \text{Product Property of Logarithms}\\
&= \ln\left|\frac{1}{\sin\theta}\cdot\frac{\sin\theta}{\cos\theta}\right| && \text{Reciprocal and quotient identities}\\
&= \ln\left|\frac{1}{\cos\theta}\right| && \text{Simplify.}\\
&= \ln|\sec\theta| && \text{Reciprocal identity}
\end{aligned}$$

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Rewrite $\ln|\sec x| + \ln|\sin x|$ as a single logarithm and simplify the result.

ALGEBRA HELP Recall that for positive real numbers u and v,

$$\ln u + \ln v = \ln(uv).$$

To review the properties of logarithms, see Section 3.3.

Summarize (Section 5.1)

1. State the fundamental trigonometric identities *(page 348)*.
2. Explain how to use the fundamental trigonometric identities to evaluate trigonometric functions, simplify trigonometric expressions, and rewrite trigonometric expressions *(pages 349–352)*. For examples of these concepts, see Examples 1–9.

5.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blank to complete the trigonometric identity.

1. $\dfrac{\sin u}{\cos u} =$ ______
2. $\dfrac{1}{\sin u} =$ ______
3. $\dfrac{1}{\tan u} =$ ______
4. $\sec\left(\dfrac{\pi}{2} - u\right) =$ ______
5. $\sin^2 u + \cos^2 u =$ ______
6. $\sin(-u) =$ ______

Skills and Applications

Using Identities to Evaluate a Function In Exercises 7–12, use the given conditions to find the values of all six trigonometric functions.

7. $\sec x = -\frac{5}{2}$, $\tan x < 0$
8. $\csc x = -\frac{7}{6}$, $\tan x > 0$
9. $\sin\theta = -\frac{3}{4}$, $\cos\theta > 0$
10. $\cos\theta = \frac{2}{3}$, $\sin\theta < 0$
11. $\tan x = \frac{2}{3}$, $\cos x > 0$
12. $\cot x = \frac{7}{4}$, $\sin x < 0$

Matching Trigonometric Expressions In Exercises 13–18, match the trigonometric expression with its simplified form.

(a) $\csc x$ (b) -1 (c) 1
(d) $\sin x\tan x$ (e) $\sec^2 x$ (f) $\sec x$

13. $\sec x\cos x$
14. $\cot^2 x - \csc^2 x$
15. $\cos x(1 + \tan^2 x)$
16. $\cot x\sec x$
17. $\dfrac{\sec^2 x - 1}{\sin^2 x}$
18. $\dfrac{\cos^2[(\pi/2) - x]}{\cos x}$

Simplifying a Trigonometric Expression In Exercises 19–22, use the fundamental identities to simplify the expression. (There is more than one correct form of each answer).

19. $\dfrac{\tan\theta\cot\theta}{\sec\theta}$
20. $\cos\left(\dfrac{\pi}{2} - x\right)\sec x$
21. $\tan^2 x - \tan^2 x\sin^2 x$
22. $\sin^2 x\sec^2 x - \sin^2 x$

Factoring a Trigonometric Expression In Exercises 23–32, factor the expression. Use the fundamental identities to simplify, if necessary. (There is more than one correct form of each answer.)

23. $\dfrac{\sec^2 x - 1}{\sec x - 1}$
24. $\dfrac{\cos x - 2}{\cos^2 x - 4}$
25. $1 - 2\cos^2 x + \cos^4 x$
26. $\sec^4 x - \tan^4 x$
27. $\cot^3 x + \cot^2 x + \cot x + 1$
28. $\sec^3 x - \sec^2 x - \sec x + 1$
29. $3\sin^2 x - 5\sin x - 2$
30. $6\cos^2 x + 5\cos x - 6$
31. $\cot^2 x + \csc x - 1$
32. $\sin^2 x + 3\cos x + 3$

Simplifying a Trigonometric Expression In Exercises 33–40, use the fundamental identities to simplify the expression. (There is more than one correct form of each answer.)

33. $\tan\theta\csc\theta$
34. $\tan(-x)\cos x$
35. $\sin\phi(\csc\phi - \sin\phi)$
36. $\cos x(\sec x - \cos x)$
37. $\sin\beta\tan\beta + \cos\beta$
38. $\cot u\sin u + \tan u\cos u$
39. $\dfrac{1 - \sin^2 x}{\csc^2 x - 1}$
40. $\dfrac{\cos^2 y}{1 - \sin y}$

Multiplying Trigonometric Expressions In Exercises 41 and 42, perform the multiplication and use the fundamental identities to simplify. (There is more than one correct form of each answer.)

41. $(\sin x + \cos x)^2$
42. $(2\csc x + 2)(2\csc x - 2)$

Adding or Subtracting Trigonometric Expressions In Exercises 43–48, perform the addition or subtraction and use the fundamental identities to simplify. (There is more than one correct form of each answer.)

43. $\dfrac{1}{1+\cos x} + \dfrac{1}{1-\cos x}$
44. $\dfrac{1}{\sec x + 1} - \dfrac{1}{\sec x - 1}$
45. $\dfrac{\cos x}{1+\sin x} - \dfrac{\cos x}{1-\sin x}$
46. $\dfrac{\sin x}{1+\cos x} + \dfrac{\sin x}{1-\cos x}$
47. $\tan x - \dfrac{\sec^2 x}{\tan x}$
48. $\dfrac{\cos x}{1+\sin x} + \dfrac{1+\sin x}{\cos x}$

Rewriting a Trigonometric Expression In Exercises 49 and 50, rewrite the expression so that it is *not* in fractional form. (There is more than one correct form of each answer.)

49. $\dfrac{\sin^2 y}{1 - \cos y}$
50. $\dfrac{5}{\tan x + \sec x}$

Trigonometric Functions and Expressions **In Exercises 51 and 52, use a graphing utility to determine which of the six trigonometric functions is equal to the expression. Verify your answer algebraically.**

51. $\dfrac{\tan x + 1}{\sec x + \csc x}$ **52.** $\dfrac{1}{\sin x}\left(\dfrac{1}{\cos x} - \cos x\right)$

Trigonometric Substitution **In Exercises 53–56, use the trigonometric substitution to write the algebraic expression as a trigonometric function of θ, where $0 < \theta < \pi/2$.**

53. $\sqrt{9 - x^2}$, $x = 3\cos\theta$

54. $\sqrt{49 - x^2}$, $x = 7\sin\theta$

55. $\sqrt{x^2 - 4}$, $x = 2\sec\theta$

56. $\sqrt{9x^2 + 25}$, $3x = 5\tan\theta$

Trigonometric Substitution **In Exercises 57 and 58, use the trigonometric substitution to write the algebraic equation as a trigonometric equation of θ, where $-\pi/2 < \theta < \pi/2$. Then find $\sin\theta$ and $\cos\theta$.**

57. $\sqrt{2} = \sqrt{4 - x^2}$, $x = 2\sin\theta$

58. $5\sqrt{3} = \sqrt{100 - x^2}$, $x = 10\cos\theta$

Solving a Trigonometric Equation **In Exercises 59 and 60, use a graphing utility to solve the equation for θ, where $0 \le \theta < 2\pi$.**

59. $\sin\theta = \sqrt{1 - \cos^2\theta}$ **60.** $\sec\theta = \sqrt{1 + \tan^2\theta}$

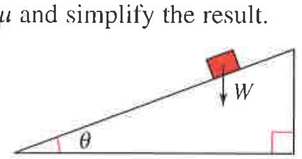
Rewriting a Logarithmic Expression **In Exercises 61–64, rewrite the expression as a single logarithm and simplify the result.**

61. $\ln|\sin x| + \ln|\cot x|$ **62.** $\ln|\cos x| - \ln|\sin x|$

63. $\ln|\tan t| - \ln(1 - \cos^2 t)$

64. $\ln(\cos^2 t) + \ln(1 + \tan^2 t)$

65. Friction The forces acting on an object weighing W units on an inclined plane positioned at an angle of θ with the horizontal (see figure) are modeled by $\mu W\cos\theta = W\sin\theta$, where μ is the coefficient of friction. Solve the equation for μ and simplify the result.



chaoss/Shutterstock.com

66. Rate of Change The rate of change of the function $f(x) = \sec x + \cos x$ is given by the expression $\sec x\tan x - \sin x$. Show that this expression can also be written as $\sin x\tan^2 x$.

Exploration

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.**

67. The quotient identities and reciprocal identities can be used to write any trigonometric function in terms of sine and cosine.

68. A cofunction identity can transform a tangent function into a cosecant function.

Analyzing Trigonometric Functions **In Exercises 69 and 70, fill in the blanks. (*Note:* The notation $x \to c^+$ indicates that x approaches c from the right and $x \to c^-$ indicates that x approaches c from the left.)**

69. As $x \to \left(\dfrac{\pi}{2}\right)^-$, $\tan x \to$ ▭ and $\cot x \to$ ▭.

70. As $x \to \pi^+$, $\sin x \to$ ▭ and $\csc x \to$ ▭.

71. Error Analysis Describe the error.

$$\frac{\sin\theta}{\cos(-\theta)} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta$$

72. Trigonometric Substitution Use the trigonometric substitution $u = a\tan\theta$, where $-\pi/2 < \theta < \pi/2$ and $a > 0$, to simplify the expression $\sqrt{a^2 + u^2}$.

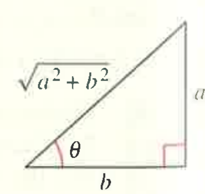
73. Writing Trigonometric Functions in Terms of Sine Write each of the other trigonometric functions of θ in terms of $\sin\theta$.

74. HOW DO YOU SEE IT? Explain how to use the figure to derive the Pythagorean identities

$\sin^2\theta + \cos^2\theta = 1$,

$1 + \tan^2\theta = \sec^2\theta$,

and $1 + \cot^2\theta = \csc^2\theta$.



Discuss how to remember these identities and other fundamental trigonometric identities.

75. Rewriting a Trigonometric Expression Rewrite the expression below in terms of $\sin\theta$ and $\cos\theta$.

$$\frac{\sec\theta(1 + \tan\theta)}{\sec\theta + \csc\theta}$$

5.2 Verifying Trigonometric Identities

Trigonometric identities enable you to rewrite trigonometric equations that model real-life situations. For example, in Exercise 62 on page 361, trigonometric identities can help you simplify an equation that models the length of a shadow cast by a gnomon (a device used to tell time).

■ **Verify trigonometric identities.**

Verifying Trigonometric Identities

In this section, you will study techniques for verifying trigonometric identities. In the next section, you will study techniques for solving trigonometric equations. The key to both verifying identities *and* solving equations is your ability to use the fundamental identities and the rules of algebra to rewrite trigonometric expressions.

Remember that a *conditional equation* is an equation that is true for only some of the values in the domain of the variable. For example, the conditional equation

$\sin x = 0$ Conditional equation

is true only for

$x = n\pi$

where n is an integer. When you are finding the values of the variable for which the equation is true, you are *solving* the equation.

On the other hand, an equation that is true for all real values in the domain of the variable is an *identity*. For example, the familiar equation

$\sin^2 x = 1 - \cos^2 x$ Identity

is true for all real numbers x. So, it is an identity.

Although there are similarities, verifying that a trigonometric equation is an identity is quite different from solving an equation. There is no well-defined set of rules to follow in verifying trigonometric identities, the process is best learned through practice.

Guidelines for Verifying Trigonometric Identities

1. Work with one side of the equation at a time. It is often better to work with the more complicated side first.
2. Look for opportunities to factor an expression, add fractions, square a binomial, or create a monomial denominator.
3. Look for opportunities to use the fundamental identities. Note which functions are in the final expression you want. Sines and cosines pair up well, as do secants and tangents, and cosecants and cotangents.
4. When the preceding guidelines do not help, try converting all terms to sines and cosines.
5. Always try *something*. Even making an attempt that leads to a dead end can provide insight.

Verifying trigonometric identities is a useful process when you need to convert a trigonometric expression into a form that is more useful algebraically. When you verify an identity, you cannot *assume* that the two sides of the equation are equal because you are trying to verify that they *are* equal. As a result, when verifying identities, you cannot use operations such as adding the same quantity to each side of the equation or cross multiplication.

Leonard Zhukovsky/Shutterstock.com