REMARK Recall from Section 4.8 that in air navigation, bearings are measured in degrees clockwise from north.

EXAMPLE 11 Using Vectors to Find Speed and Direction

An airplane travels at a speed of 500 miles per hour with a bearing of 330° at a fixed altitude with a negligible wind velocity, as shown in Figure 6.24(a). (Note that a bearing of 330° corresponds to a direction angle of 120°.) The airplane encounters a wind with a velocity of 70 miles per hour in the direction N 45° E, as shown in Figure 6.24(b). What are the resultant speed and true direction of the airplane?

Solution Using Figure 6.24, the velocity of the airplane (alone) is

$$\mathbf{v}_1 = 500\langle \cos 120°, \sin 120° \rangle = \langle -250, 250\sqrt{3} \rangle$$

and the velocity of the wind is

$$\mathbf{v}_2 = 70\langle \cos 45°, \sin 45° \rangle = \langle 35\sqrt{2}, 35\sqrt{2} \rangle.$$

So, the velocity of the airplane (in the wind) is

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$
$$= \langle -250 + 35\sqrt{2}, 250\sqrt{3} + 35\sqrt{2} \rangle$$
$$\approx \langle -200.5, 482.5 \rangle$$

and the resultant speed of the airplane is

$$\|\mathbf{v}\| \approx \sqrt{(-200.5)^2 + (482.5)^2} \approx 522.5 \text{ miles per hour.}$$

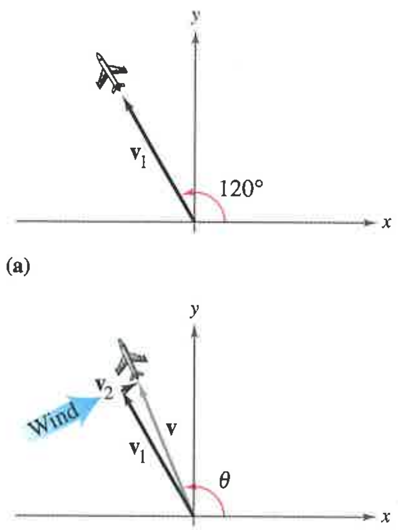
To find the direction angle θ of the flight path, you have

$$\tan \theta \approx \frac{482.5}{-200.5} \approx -2.4065.$$

The flight path lies in Quadrant II, so θ lies in Quadrant II, and its reference angle is

$$\theta' \approx |\arctan(-2.4065)| \approx |-1.1770 \text{ radians}| \approx |-67.44°| = 67.44°.$$

So, the direction angle is $\theta \approx 180° - 67.44° = 112.56°$, and the true direction of the airplane is approximately $270° + (180° - 112.56°) = 337.44°$.



(a)

(b)

Figure 6.24

Pilots can take advantage of fast-moving air currents called jet streams to decrease travel time.

✓ Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com

Repeat Example 11 for an airplane traveling at a speed of 450 miles per hour with a bearing of 300° that encounters a wind with a velocity of 40 miles per hour in the direction N 30° E.

Summarize (Section 6.3)

1. Explain how to represent a vector as a directed line segment *(page 416)*. For an example involving vectors represented as directed line segments, see Example 1.
2. Explain how to find the component form of a vector *(page 417)*. For an example of finding the component form of a vector, see Example 2.
3. Explain how to perform basic vector operations *(page 418)*. For an example of performing basic vector operations, see Example 3.
4. Explain how to write a vector as a linear combination of unit vectors *(page 420)*. For examples involving unit vectors, see Examples 5–7.
5. Explain how to find the direction angle of a vector *(page 422)*. For an example of finding direction angles of vectors, see Example 8.
6. Describe real-life applications of vectors *(pages 423 and 424, Examples 9–11)*.

6.3 Exercises

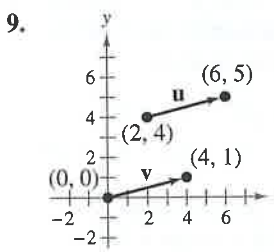
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

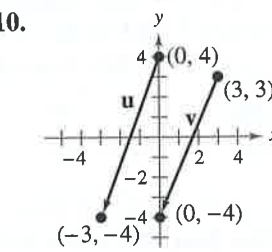
Vocabulary: Fill in the blanks.

1. You can use a _______ _______ _______ to represent a quantity that involves both magnitude and direction.
2. The directed line segment $\overrightarrow{PQ}$ has _______ point P and _______ point Q.
3. The set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ is a _______ $\mathbf{v}$ in the plane.
4. Two vectors are equivalent when they have the same _______ and the same _______.
5. The directed line segment whose initial point is the origin is in _______ _______.
6. A vector that has a magnitude of 1 is a _______ _______.
7. The two basic vector operations are scalar _______ and vector _______.
8. The vector sum $v_1\mathbf{i} + v_2\mathbf{j}$ is a _______ _______ of the vectors $\mathbf{i}$ and $\mathbf{j}$, and the scalars v_1 and v_2 are the _______ and _______ components of $\mathbf{v}$, respectively.

Skills and Applications

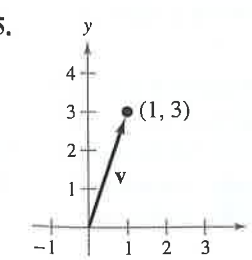
Determining Whether Two Vectors Are Equivalent In Exercises 9–14, determine whether u and v are equivalent. Explain.

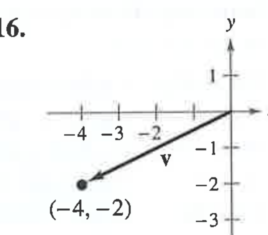
9.


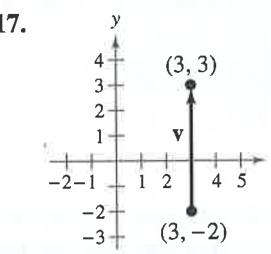
10.


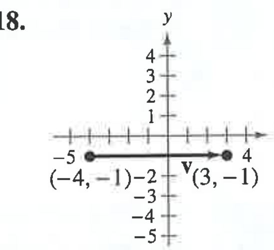
Vector	Initial Point	Terminal Point
11. u	(2, 2)	(−1, 4)
v	(−3, −1)	(−5, 2)
12. u	(2, 0)	(7, 4)
v	(−8, 1)	(2, 9)
13. u	(2, −1)	(5, −10)
v	(6, 1)	(9, −8)
14. u	(8, 1)	(13, −1)
v	(−2, 4)	(−7, 6)

Finding the Component Form of a Vector In Exercises 15–24, find the component form and magnitude of the vector v.

15.


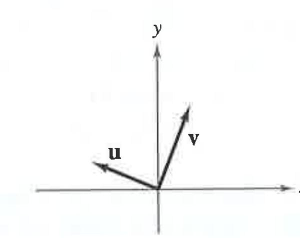
16.


17.


18.


	Initial Point	Terminal Point
19.	(−3, −5)	(−11, 1)
20.	(−2, 7)	(5, −17)
21.	(1, 3)	(−8, −9)
22.	(17, −5)	(9, 3)
23.	(−1, 5)	(15, −21)
24.	(−3, 11)	(9, 40)

Sketching the Graph of a Vector In Exercises 25–30, use the figure to sketch a graph of the specified vector. To print an enlarged copy of the graph, go to *MathGraphs.com*.



25. $-\mathbf{v}$
26. $5\mathbf{v}$
27. $\mathbf{u} + \mathbf{v}$
28. $\mathbf{u} + 2\mathbf{v}$
29. $\mathbf{u} - \mathbf{v}$
30. $\mathbf{v} - \frac{1}{2}\mathbf{u}$

Vector Operations **In Exercises 31–36, find (a) u + v, (b) u − v, and (c) 2u − 3v. Then sketch each resultant vector.**

31. $\mathbf{u} = \langle 2, 1\rangle$, $\mathbf{v} = \langle 1, 3\rangle$
32. $\mathbf{u} = \langle 2, 3\rangle$, $\mathbf{v} = \langle 4, 0\rangle$
33. $\mathbf{u} = \langle -5, 3\rangle$, $\mathbf{v} = \langle 0, 0\rangle$
34. $\mathbf{u} = \langle 0, 0\rangle$, $\mathbf{v} = \langle 2, 1\rangle$
35. $\mathbf{u} = \langle 0, -7\rangle$, $\mathbf{v} = \langle 1, -2\rangle$
36. $\mathbf{u} = \langle -3, 1\rangle$, $\mathbf{v} = \langle 2, -5\rangle$

Finding the Magnitude of a Scalar Multiple **In Exercises 37–40, find the magnitude of the scalar multiple, where $\mathbf{u} = \langle 2, 0\rangle$ and $\mathbf{v} = \langle -3, 6\rangle$.**

37. $\|5\mathbf{u}\|$
38. $\|4\mathbf{v}\|$
39. $\|-3\mathbf{v}\|$
40. $\left\|-\frac{3}{4}\mathbf{u}\right\|$

Finding a Unit Vector **In Exercises 41–46, find a unit vector u in the direction of v. Verify that $\|\mathbf{u}\| = 1$.**

41. $\mathbf{v} = \langle 3, 0\rangle$
42. $\mathbf{v} = \langle 0, -2\rangle$
43. $\mathbf{v} = \langle -2, 2\rangle$
44. $\mathbf{v} = \langle -5, 12\rangle$
45. $\mathbf{v} = \langle 1, -6\rangle$
46. $\mathbf{v} = \langle -8, -4\rangle$

Finding a Vector **In Exercises 47–50, find the vector v with the given magnitude and the same direction as u.**

47. $\|\mathbf{v}\| = 10$, $\mathbf{u} = \langle -3, 4\rangle$
48. $\|\mathbf{v}\| = 3$, $\mathbf{u} = \langle -12, -5\rangle$
49. $\|\mathbf{v}\| = 9$, $\mathbf{u} = \langle 2, 5\rangle$
50. $\|\mathbf{v}\| = 8$, $\mathbf{u} = \langle 3, 3\rangle$

Writing a Linear Combination of Unit Vectors **In Exercises 51–54, the initial and terminal points of a vector are given. Write the vector as a linear combination of the standard unit vectors i and j.**

	Initial Point	Terminal Point
51.	$(-2, 1)$	$(3, -2)$
52.	$(0, -2)$	$(3, 6)$
53.	$(0, 1)$	$(-6, 4)$
54.	$(2, 3)$	$(-1, -5)$

Vector Operations **In Exercises 55–60, find the component form of v and sketch the specified vector operations geometrically, where $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j}$.**

55. $\mathbf{v} = \frac{3}{2}\mathbf{u}$
56. $\mathbf{v} = \frac{3}{4}\mathbf{w}$
57. $\mathbf{v} = \mathbf{u} + 2\mathbf{w}$
58. $\mathbf{v} = -\mathbf{u} + \mathbf{w}$
59. $\mathbf{v} = \mathbf{u} - 2\mathbf{w}$
60. $\mathbf{v} = \frac{1}{2}(3\mathbf{u} + \mathbf{w})$

Finding the Direction Angle of a Vector **In Exercises 61–64, find the magnitude and direction angle of the vector v.**

61. $\mathbf{v} = 6\mathbf{i} - 6\mathbf{j}$
62. $\mathbf{v} = -5\mathbf{i} + 4\mathbf{j}$
63. $\mathbf{v} = 3(\cos 60°\mathbf{i} + \sin 60°\mathbf{j})$
64. $\mathbf{v} = 8(\cos 135°\mathbf{i} + \sin 135°\mathbf{j})$

Finding the Component Form of a Vector **In Exercises 65–70, find the component form of v given its magnitude and the angle it makes with the positive x-axis. Then sketch v.**

	Magnitude	Angle
65.	$\|\mathbf{v}\| = 3$	$\theta = 0°$
66.	$\|\mathbf{v}\| = 4\sqrt{3}$	$\theta = 90°$
67.	$\|\mathbf{v}\| = \frac{7}{2}$	$\theta = 150°$
68.	$\|\mathbf{v}\| = 2\sqrt{3}$	$\theta = 45°$
69.	$\|\mathbf{v}\| = 3$	$\mathbf{v}$ in the direction $3\mathbf{i} + 4\mathbf{j}$
70.	$\|\mathbf{v}\| = 2$	$\mathbf{v}$ in the direction $\mathbf{i} + 3\mathbf{j}$

Finding the Component Form of a Vector **In Exercises 71 and 72, find the component form of the sum of u and v with direction angles θ_u and θ_v.**

71. $\|\mathbf{u}\| = 4$, $\theta_u = 60°$
$\|\mathbf{v}\| = 4$, $\theta_v = 90°$
72. $\|\mathbf{u}\| = 20$, $\theta_u = 45°$
$\|\mathbf{v}\| = 50$, $\theta_v = 180°$

Using the Law of Cosines **In Exercises 73 and 74, use the Law of Cosines to find the angle α between the vectors. (Assume $0° \le \alpha \le 180°$.)**

73. $\mathbf{v} = \mathbf{i} + \mathbf{j}$, $\mathbf{w} = 2\mathbf{i} - 2\mathbf{j}$
74. $\mathbf{v} = \mathbf{i} + 2\mathbf{j}$, $\mathbf{w} = 2\mathbf{i} - \mathbf{j}$

Resultant Force **In Exercises 75 and 76, find the angle between the forces given the magnitude of their resultant. (*Hint:* Write force 1 as a vector in the direction of the positive x-axis and force 2 as a vector at an angle θ with the positive x-axis.)**

	Force 1	Force 2	Resultant Force
75.	45 pounds	60 pounds	90 pounds
76.	3000 pounds	1000 pounds	3750 pounds

77. **Velocity** A gun with a muzzle velocity of 1200 feet per second is fired at an angle of 6° above the horizontal. Find the vertical and horizontal components of the velocity.

78. **Velocity** Pitcher Aroldis Chapman threw a pitch with a recorded velocity of 105 miles per hour. Assuming he threw the pitch at an angle of 3.5° below the horizontal, find the vertical and horizontal components of the velocity. *(Source: Guinness World Records)*

79. **Resultant Force** Forces with magnitudes of 125 newtons and 300 newtons act on a hook (see figure). The angle between the two forces is 45°. Find the direction and magnitude of the resultant of these forces. (*Hint:* Write the vector representing each force in component form, then add the vectors.)

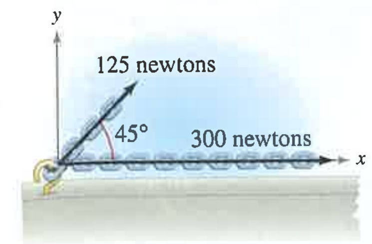


Figure for 79

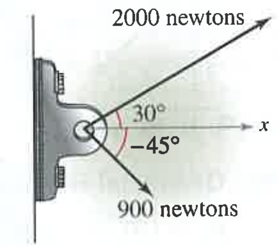


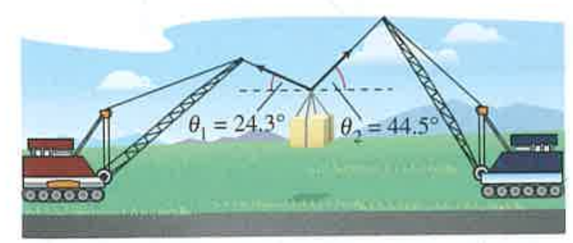
Figure for 80

80. **Resultant Force** Forces with magnitudes of 2000 newtons and 900 newtons act on a machine part at angles of 30° and −45°, respectively, with the positive x-axis (see figure). Find the direction and magnitude of the resultant of these forces.

81. **Resultant Force** Three forces with magnitudes of 75 pounds, 100 pounds, and 125 pounds act on an object at angles of 30°, 45°, and 120°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant of these forces.

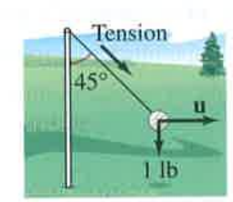
82. **Resultant Force** Three forces with magnitudes of 70 pounds, 40 pounds, and 60 pounds act on an object at angles of −30°, 45°, and 135°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant of these forces.

83. **Cable Tension** The cranes shown in the figure are lifting an object that weighs 20,240 pounds. Find the tension (in pounds) in the cable of each crane.

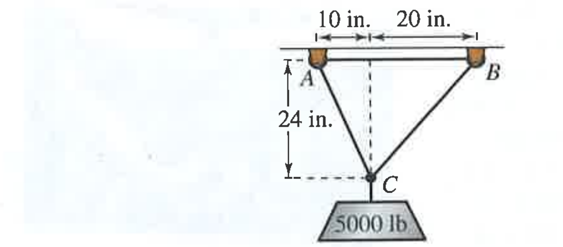


84. **Cable Tension** Repeat Exercise 83 for $\theta_1 = 35.6°$ and $\theta_2 = 40.4°$.

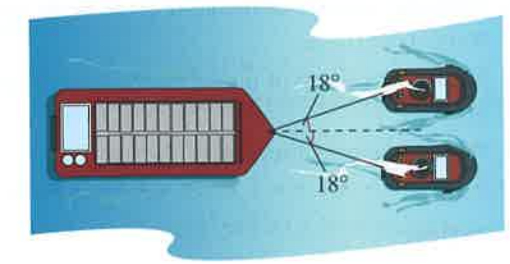
85. **Rope Tension** A tetherball weighing 1 pound is pulled outward from the pole by a horizontal force u until the rope makes a 45° angle with the pole (see figure). Determine the resulting tension (in pounds) in the rope and the magnitude of u.



86. **Physics** Use the figure to determine the tension (in pounds) in each cable supporting the load.



87. **Tow Line Tension** Two tugboats are towing a loaded barge and the magnitude of the resultant is 6000 pounds directed along the axis of the barge (see figure). Find the tension (in pounds) in the tow lines when they each make an 18° angle with the axis of the barge.

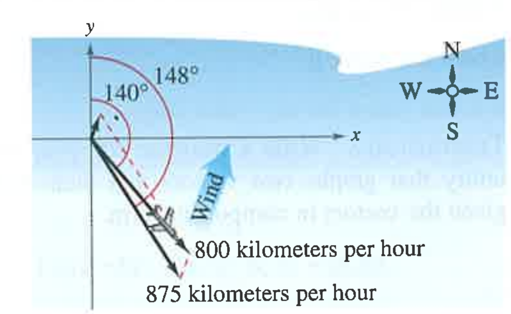


88. **Rope Tension** To carry a 100-pound cylindrical weight, two people lift on the ends of short ropes that are tied to an eyelet on the top center of the cylinder. Each rope makes a 20° angle with the vertical. Draw a diagram that gives a visual representation of the problem. Then find the tension (in pounds) in the ropes.

Inclined Ramp **In Exercises 89–92, a force of F pounds is required to pull an object weighing W pounds up a ramp inclined at θ degrees from the horizontal.**

89. Find F when $W = 100$ pounds and $\theta = 12°$.
90. Find W when $F = 600$ pounds and $\theta = 14°$.
91. Find θ when $F = 5000$ pounds and $W = 15{,}000$ pounds.
92. Find F when $W = 5000$ pounds and $\theta = 26°$.

93. **Air Navigation** An airplane travels in the direction of 148° with an airspeed of 875 kilometers per hour. Due to the wind, its groundspeed and direction are 800 kilometers per hour and 140°, respectively (see figure). Find the direction and speed of the wind.



94. Air Navigation A commercial jet travels from Miami to Seattle. The jet's velocity with respect to the air is 580 miles per hour, and its bearing is 332°. The jet encounters a wind with a velocity of 60 miles per hour from the southwest.

(a) Draw a diagram that gives a visual representation of the problem.

(b) Write the velocity of the wind as a vector in component form.

(c) Write the velocity of the jet relative to the air in component form.

(d) What is the speed of the jet with respect to the ground?

(e) What is the true direction of the jet?

Exploration

True or False? **In Exercises 95–98, determine whether the statement is true or false. Justify your answer.**

95. If **u** and **v** have the same magnitude and direction, then **u** and **v** are equivalent.

96. If **u** is a unit vector in the direction of **v**, then $\mathbf{v} = \|\mathbf{v}\|\mathbf{u}$.

97. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \mathbf{0}$, then $a = -b$.

98. If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then $a^2 + b^2 = 1$.

99. Error Analysis Describe the error in finding the component form of the vector **u** that has initial point $(-3, 4)$ and terminal point $(6, -1)$.

The components are $u_1 = -3 - 6 = -9$ and $u_2 = 4 - (-1) = 5$. So, $\mathbf{u} = \langle -9, 5 \rangle$. ✗

100. Error Analysis Describe the error in finding the direction angle θ of the vector $\mathbf{v} = -5\mathbf{i} + 8\mathbf{j}$.

Because $\tan\theta = \frac{b}{a} = \frac{8}{-5}$, the reference angle is $\theta' = \left|\arctan\left(-\frac{8}{5}\right)\right| \approx |-57.99°| = 57.99°$ and $\theta \approx 360° - 57.99° = 302.01°$. ✗

101. Proof Prove that

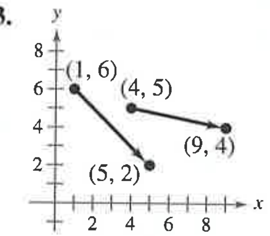
$$(\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$$

is a unit vector for any value of θ.

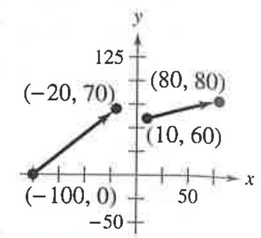
102. Technology Write a program for your graphing utility that graphs two vectors and their difference given the vectors in component form.

Finding the Difference of Two Vectors **In Exercises 103 and 104, use the program in Exercise 102 to find the difference of the vectors shown in the figure.**

103.



104.



105. Graphical Reasoning Consider two forces

$$\mathbf{F}_1 = \langle 10, 0 \rangle \quad \text{and} \quad \mathbf{F}_2 = 5\langle \cos\theta, \sin\theta \rangle.$$

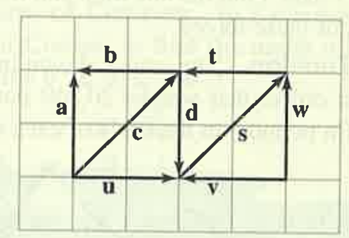
(a) Find $\|\mathbf{F}_1 + \mathbf{F}_2\|$ as a function of θ.

(b) Use a graphing utility to graph the function in part (a) for $0 \le \theta < 2\pi$.

(c) Use the graph in part (b) to determine the range of the function. What is its maximum, and for what value of θ does it occur? What is its minimum, and for what value of θ does it occur?

(d) Explain why the magnitude of the resultant is never 0.

106. HOW DO YOU SEE IT? Use the figure to determine whether each statement is true or false. Justify your answer.



(a) $\mathbf{a} = -\mathbf{d}$ (b) $\mathbf{c} = \mathbf{s}$

(c) $\mathbf{a} + \mathbf{u} = \mathbf{c}$ (d) $\mathbf{v} + \mathbf{w} = -\mathbf{s}$

(e) $\mathbf{a} + \mathbf{w} = -2\mathbf{d}$ (f) $\mathbf{a} + \mathbf{d} = \mathbf{0}$

(g) $\mathbf{u} - \mathbf{v} = -2(\mathbf{b} + \mathbf{t})$ (h) $\mathbf{t} - \mathbf{w} = \mathbf{b} - \mathbf{a}$

107. Writing Give geometric descriptions of (a) vector addition and (b) scalar multiplication.

108. Writing Identify the quantity as a scalar or as a vector. Explain.

(a) The muzzle velocity of a bullet

(b) The price of a company's stock

(c) The air temperature in a room

(d) The weight of an automobile

iStockphoto.com/Vladimir Maravic

6.4 Vectors and Dot Products

The dot product of two vectors has many real-life applications. For example, in Exercise 74 on page 436, you will use the dot product to find the force necessary to keep a sport utility vehicle from rolling down a hill.

- **Find the dot product of two vectors and use the properties of the dot product.**
- **Find the angle between two vectors and determine whether two vectors are orthogonal.**
- **Write a vector as the sum of two vector components.**
- **Use vectors to determine the work done by a force.**

The Dot Product of Two Vectors

So far, you have studied two vector operations—vector addition and multiplication by a scalar—each of which yields another vector. In this section, you will study a third vector operation, the **dot product.** This operation yields a scalar, rather than a vector.

Definition of the Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2$.

Properties of the Dot Product

Let **u**, **v**, and **w** be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $\mathbf{0} \cdot \mathbf{v} = 0$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$
5. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$

For proofs of the properties of the dot product, see Proofs in Mathematics on page 464.

REMARK In Example 1, be sure you see that the dot product of two vectors is a scalar (a real number), not a vector. Moreover, notice that the dot product can be positive, zero, or negative.

EXAMPLE 1 Finding Dot Products

a. $\langle 4, 5 \rangle \cdot \langle 2, 3 \rangle = 4(2) + 5(3)$
$= 8 + 15$
$= 23$

b. $\langle 2, -1 \rangle \cdot \langle 1, 2 \rangle = 2(1) + (-1)(2)$
$= 2 - 2$
$= 0$

c. $\langle 0, 3 \rangle \cdot \langle 4, -2 \rangle = 0(4) + 3(-2)$
$= 0 - 6$
$= -6$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com*

Find each dot product.

a. $\langle 3, 4 \rangle \cdot \langle 2, -3 \rangle$ **b.** $\langle -3, -5 \rangle \cdot \langle 1, -8 \rangle$ **c.** $\langle -6, 5 \rangle \cdot \langle 5, 6 \rangle$

Anthony Berenyi/Shutterstock.com