8.4 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Vocabulary: Fill in the blanks.

1. Both $\det(A)$ and $|A|$ represent the _____ of the matrix A.
2. The _____ M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by deleting the ith row and jth column of the square matrix A.
3. The _____ C_{ij} of the entry a_{ij} of the square matrix A is given by $(-1)^{i+j}M_{ij}$.
4. The method of finding the determinant of a matrix of dimension 2×2 or greater is called _____ by _____.

Skills and Applications

Finding the Determinant of a Matrix In Exercises 5–22, find the determinant of the matrix.

5. $[4]$

6. $[-10]$

7. $\begin{bmatrix} 8 & 4 \\ 2 & 3 \end{bmatrix}$

8. $\begin{bmatrix} -9 & 0 \\ 6 & -2 \end{bmatrix}$

9. $\begin{bmatrix} 6 & -3 \\ -5 & 2 \end{bmatrix}$

10. $\begin{bmatrix} 3 & -3 \\ 4 & -8 \end{bmatrix}$

11. $\begin{bmatrix} -7 & 0 \\ 3 & 0 \end{bmatrix}$

12. $\begin{bmatrix} 4 & -3 \\ 0 & 0 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$

14. $\begin{bmatrix} 2 & -3 \\ -6 & 9 \end{bmatrix}$

15. $\begin{bmatrix} -3 & -2 \\ -6 & -4 \end{bmatrix}$

16. $\begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix}$

17. $\begin{bmatrix} -2 & -7 \\ -3 & 1 \end{bmatrix}$

18. $\begin{bmatrix} 2 & -5 \\ -4 & -1 \end{bmatrix}$

19. $\begin{bmatrix} -7 & 6 \\ 0.5 & 3 \end{bmatrix}$

20. $\begin{bmatrix} 0 & 2.5 \\ -3 & 2 \end{bmatrix}$

21. $\begin{bmatrix} -\frac{1}{2} & \frac{1}{3} \\ -6 & \frac{1}{3} \end{bmatrix}$

22. $\begin{bmatrix} \frac{2}{3} & -\frac{4}{3} \\ -1 & \frac{1}{3} \end{bmatrix}$

Using a Graphing Utility In Exercises 23–28, use the matrix capabilities of a graphing utility to find the determinant of the matrix.

23. $\begin{bmatrix} 3 & 4 \\ -2 & 1 \end{bmatrix}$

24. $\begin{bmatrix} 5 & -9 \\ 7 & 16 \end{bmatrix}$

25. $\begin{bmatrix} 19 & 20 \\ 43 & -56 \end{bmatrix}$

26. $\begin{bmatrix} 101 & 197 \\ -253 & 172 \end{bmatrix}$

27. $\begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ -\frac{3}{10} & \frac{1}{5} \end{bmatrix}$

28. $\begin{bmatrix} 0.1 & 0.1 \\ 7.5 & 6.2 \end{bmatrix}$

Finding the Minors and Cofactors of a Matrix In Exercises 29–34, find all the (a) minors and (b) cofactors of the matrix.

29. $\begin{bmatrix} 4 & 5 \\ 3 & -6 \end{bmatrix}$

30. $\begin{bmatrix} 0 & 10 \\ 3 & -4 \end{bmatrix}$

31. $\begin{bmatrix} 4 & 0 & 2 \\ -3 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

32. $\begin{bmatrix} 1 & -1 & 0 \\ 3 & 2 & 5 \\ 4 & -6 & 4 \end{bmatrix}$

33. $\begin{bmatrix} -4 & 6 & 3 \\ 7 & -2 & 8 \\ 1 & 0 & -5 \end{bmatrix}$

34. $\begin{bmatrix} -2 & 9 & 4 \\ 7 & -6 & 0 \\ 6 & 7 & -6 \end{bmatrix}$

Finding the Determinant of a Matrix In Exercises 35–44, find the determinant of the matrix. Expand by cofactors using the indicated row or column.

35. $\begin{bmatrix} 2 & 5 \\ 6 & -3 \end{bmatrix}$
(a) Row 1
(b) Column 1

36. $\begin{bmatrix} 7 & -1 \\ -4 & 10 \end{bmatrix}$
(a) Row 2
(b) Column 2

37. $\begin{bmatrix} 5 & 0 & -3 \\ 0 & 12 & 4 \\ 1 & 6 & 3 \end{bmatrix}$
(a) Row 2
(b) Column 2

38. $\begin{bmatrix} 3 & -2 & 5 \\ 1 & 0 & 3 \\ 0 & 4 & -1 \end{bmatrix}$
(a) Row 3
(b) Column 1

39. $\begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 6 \\ 2 & -3 & 1 \end{bmatrix}$
(a) Row 1
(b) Column 2

40. $\begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}$
(a) Row 2
(b) Column 3

41. $\begin{bmatrix} 6 & 0 & -3 & 5 \\ 4 & 0 & 6 & -8 \\ -1 & 0 & 7 & 4 \\ 8 & 0 & 0 & 2 \end{bmatrix}$
(a) Row 4
(b) Column 2

42. $\begin{bmatrix} 10 & 8 & 3 & -7 \\ 4 & 0 & 5 & -6 \\ 0 & 3 & 2 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
(a) Row 4
(b) Column 1

43. $\begin{bmatrix} -2 & 4 & 7 & 1 \\ 3 & 0 & 0 & 0 \\ 8 & 5 & 10 & 5 \\ 6 & 0 & 5 & 0 \end{bmatrix}$
(a) Row 2
(b) Column 4

44. $\begin{bmatrix} 7 & 0 & 0 & -6 \\ 6 & 0 & 1 & -2 \\ 1 & -2 & 3 & 2 \\ -3 & 0 & -1 & 4 \end{bmatrix}$
(a) Row 1
(b) Column 2

Finding the Determinant of a Matrix In Exercises 45–58, find the determinant of the matrix. Expand by cofactors using the row or column that appears to make the computations easiest.

45. $\begin{bmatrix} -1 & 8 & -3 \\ 0 & 3 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

46. $\begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 4 & 11 & 5 \end{bmatrix}$

47. $\begin{bmatrix} 6 & 3 & -7 \\ 0 & 0 & 0 \\ 4 & -6 & 3 \end{bmatrix}$

48. $\begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \\ -2 & 0 & 3 \end{bmatrix}$

49. $\begin{bmatrix} 2 & -1 & 0 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

50. $\begin{bmatrix} -2 & 2 & 3 \\ 1 & -1 & 0 \\ 0 & 1 & 4 \end{bmatrix}$

51. $\begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$

52. $\begin{bmatrix} 2 & -1 & 3 \\ -4 & 2 & -6 \\ 1 & 0 & 2 \end{bmatrix}$

53. $\begin{bmatrix} 2 & 6 & 0 & 2 \\ 2 & 7 & 3 & 6 \\ 1 & 0 & 0 & 1 \\ 3 & 7 & 0 & 7 \end{bmatrix}$

54. $\begin{bmatrix} 1 & 4 & 3 & 2 \\ -5 & 6 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 5 \end{bmatrix}$

55. $\begin{bmatrix} 5 & 3 & 0 & 6 \\ 4 & 6 & 4 & 12 \\ 0 & 2 & -3 & 4 \\ 0 & 1 & -2 & 2 \end{bmatrix}$

56. $\begin{bmatrix} 3 & 6 & -5 & 4 \\ -2 & 0 & 6 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 3 & -1 & -1 \end{bmatrix}$

57. $\begin{bmatrix} 3 & 2 & 4 & -1 & 5 \\ -2 & 0 & 1 & 3 & 2 \\ 1 & 0 & 0 & 4 & 0 \\ 6 & 0 & 2 & -1 & 0 \\ 3 & 0 & 5 & 1 & 0 \end{bmatrix}$

58. $\begin{bmatrix} 5 & 2 & 0 & 0 & -2 \\ 0 & 1 & 4 & 3 & \frac{1}{2} \\ 0 & 0 & 2 & 6 & 3 \\ 0 & 0 & 3 & \frac{3}{2} & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

Using a Graphing Utility In Exercises 59–62, use the matrix capabilities of a graphing utility to find the determinant.

59. $\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 8 & 1 & 6 \end{vmatrix}$

60. $\begin{vmatrix} 5 & -8 & 0 \\ 9 & 7 & 4 \\ -8 & 7 & 1 \end{vmatrix}$

61. $\begin{vmatrix} 1 & -1 & 8 & 4 \\ 2 & 6 & 0 & -4 \\ 2 & 0 & 2 & 6 \\ 0 & 2 & 8 & 0 \end{vmatrix}$

62. $\begin{vmatrix} 0 & -3 & 8 & 2 \\ 8 & 1 & -1 & 6 \\ -4 & 6 & 0 & 9 \\ -7 & 0 & 0 & 14 \end{vmatrix}$

The Determinant of a Matrix Product In Exercises 63–68, find (a) $|A|$, (b) $|B|$, (c) AB, and (d) $|AB|$.

63. $A = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

64. $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

65. $A = \begin{bmatrix} 4 & 0 \\ 3 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

66. $A = \begin{bmatrix} 5 & 4 \\ 3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 6 \\ 1 & -2 \end{bmatrix}$

67. $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

68. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 4 \\ 0 & 1 & 3 \\ 3 & -2 & 1 \end{bmatrix}$

Creating a Matrix In Exercises 69–74, create a matrix A with the given characteristics. (There are many correct answers.)

69. Dimension: 2×2, $|A| = 3$
70. Dimension: 2×2, $|A| = -5$
71. Dimension: 3×3, $|A| = -1$
72. Dimension: 3×3, $|A| = 4$
73. Dimension: 2×2, $|A| = 0$, $A \neq O$
74. Dimension: 3×3, $|A| = 0$, $A \neq O$

Verifying an Equation In Exercises 75–80, find the determinant(s) to verify the equation.

75. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = -\begin{vmatrix} y & z \\ w & x \end{vmatrix}$

76. $\begin{vmatrix} w & cx \\ y & cz \end{vmatrix} = c\begin{vmatrix} w & x \\ y & z \end{vmatrix}$

77. $\begin{vmatrix} w & x \\ y & z \end{vmatrix} = \begin{vmatrix} w & x + cw \\ y & z + cy \end{vmatrix}$

78. $\begin{vmatrix} w & x \\ cw & cx \end{vmatrix} = 0$

79. $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (y - x)(z - x)(z - y)$

80. $\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = b^2(3a + b)$

**Solving an Equation** In Exercises 81–86, solve for x.

81. $\begin{vmatrix} x & 2 \\ 1 & x \end{vmatrix} = 2$

82. $\begin{vmatrix} x & 4 \\ -1 & x \end{vmatrix} = 20$

83. $\begin{vmatrix} x+1 & 2 \\ -1 & x \end{vmatrix} = 4$

84. $\begin{vmatrix} x-2 & -1 \\ -3 & x \end{vmatrix} = 0$

85. $\begin{vmatrix} x+3 & 2 \\ 1 & x+2 \end{vmatrix} = 0$

86. $\begin{vmatrix} x+4 & -2 \\ 7 & x-5 \end{vmatrix} = 0$

• • **Entries Involving Expressions** • • • • • • • •

In Exercises 87–92, find the determinant in which the entries are functions. Determinants of this type occur when changes of variables are made in calculus.

$Let\ x = ce^{2t}$
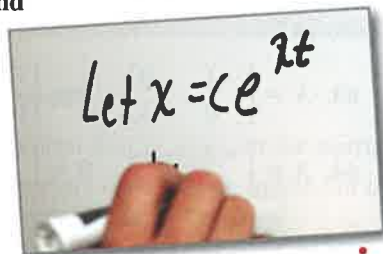
87. $\begin{vmatrix} 4u & -1 \\ -1 & 2v \end{vmatrix}$

88. $\begin{vmatrix} 3x^2 & -3y^2 \\ 1 & 1 \end{vmatrix}$

89. $\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$

90. $\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}$

91. $\begin{vmatrix} x & \ln x \\ 1 & 1/x \end{vmatrix}$

92. $\begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix}$

Exploration

True or False? In Exercises 93 and 94, determine whether the statement is true or false. Justify your answer.

93. If a square matrix has an entire row of zeros, then the determinant of the matrix is zero.

94. If the rows of a 2×2 matrix are the same, then the determinant of the matrix is zero.

95. Think About It Find square matrices A and B such that $|A + B| \neq |A| + |B|$.

96. Conjecture Consider square matrices in which the entries are consecutive integers. An example of such a matrix is

$$\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}.$$

(a) Use the matrix capabilities of a graphing utility to find the determinants of four matrices of this type. Make a conjecture based on the results.

(b) Verify your conjecture.

97. Error Analysis Describe the error.

$$\begin{vmatrix} 1 & 1 & 4 \\ 3 & 2 & 0 \\ 2 & 1 & 3 \end{vmatrix} = 3(1)\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} + 2(-1)\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$$
$$+ 0(1)\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$$
$$= 3(-1) - 2(-5) + 0$$
$$= 7$$

98. Think About It Let A be a 3×3 matrix such that $|A| = 5$. Is it possible to find $|2A|$? Explain.

Properties of Determinants In Exercises 99–101, explain why each equation is an example of the given property of determinants (A and B are square matrices). Use a graphing utility to verify the results.

99. If B is obtained from A by interchanging two rows of A or interchanging two columns of A, then $|B| = -|A|$.

(a) $\begin{vmatrix} 1 & 3 & 4 \\ -7 & 2 & -5 \\ 6 & 1 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 4 & 3 \\ -7 & -5 & 2 \\ 6 & 2 & 1 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 3 & 4 \\ -2 & 2 & 0 \\ 1 & 6 & 2 \end{vmatrix} = -\begin{vmatrix} 1 & 6 & 2 \\ -2 & 2 & 0 \\ 1 & 3 & 4 \end{vmatrix}$

100. If B is obtained from A by adding a multiple of a row of A to another row of A or by adding a multiple of a column of A to another column of A, then $|B| = |A|$.

(a) $\begin{vmatrix} 1 & -3 \\ 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -3 \\ 0 & 17 \end{vmatrix}$

(b) $\begin{vmatrix} 5 & 4 & 2 \\ 2 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 10 & -6 \\ 2 & -3 & 4 \\ 7 & 6 & 3 \end{vmatrix}$

101. If B is obtained from A by multiplying a row by a nonzero constant c or by multiplying a column by a nonzero constant c, then $|B| = c|A|$.

(a) $\begin{vmatrix} 5 & 10 \\ 2 & -3 \end{vmatrix} = 5\begin{vmatrix} 1 & 2 \\ 2 & -3 \end{vmatrix}$

(b) $\begin{vmatrix} 1 & 8 & -3 \\ 3 & -12 & 6 \\ 7 & 4 & 9 \end{vmatrix} = 12\begin{vmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ 7 & 1 & 3 \end{vmatrix}$

102. HOW DO YOU SEE IT? Explain why the determinant of each matrix is equal to zero.

(a) $\begin{bmatrix} 2 & -4 & 5 \\ 1 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & -4 & 5 & 7 \\ 2 & -2 & 3 & 1 \\ 4 & -4 & 5 & 7 \\ 6 & 1 & -3 & -3 \end{bmatrix}$

103. Conjecture A **diagonal matrix** is a square matrix in which each entry not on the main diagonal is zero. Find the determinant of each diagonal matrix. Make a conjecture based on your results.

(a) $\begin{bmatrix} 7 & 0 \\ 0 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

8.5 Applications of Matrices and Determinants

Determinants have many applications in real life. For example, in Exercise 21 on page 595, you will use a determinant to find the area of a region of forest infested with gypsy moths.

■ Use Cramer's Rule to solve systems of linear equations.
■ Use determinants to find areas of triangles.
■ Use determinants to test for collinear points and find equations of lines passing through two points.
■ Use 2×2 matrices to perform transformations in the plane and find areas of parallelograms.
■ Use matrices to encode and decode messages.

Cramer's Rule

So far, you have studied four methods for solving a system of linear equations: substitution, graphing, elimination with equations, and elimination with matrices. In this section, you will study one more method, **Cramer's Rule,** named after the Swiss mathematician Gabriel Cramer (1704–1752). This rule uses determinants to write the solution of a system of linear equations. To see how Cramer's Rule works, consider the system described at the beginning of Section 8.4, which is shown below.

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

This system has a solution

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \quad \text{and} \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

provided that

$$a_1b_2 - a_2b_1 \neq 0.$$

Each numerator and denominator in this solution can be expressed as a determinant.

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Relative to the original system, the denominators for x and y are the determinant of the *coefficient* matrix of the system. This determinant is denoted by D. The numerators for x and y are denoted by D_x and D_y, respectively, and are formed by using the column of constants as replacements for the coefficients of x and y.

Coefficient Matrix	D	D_x	D_y
$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$	$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

For example, given the system

$$\begin{cases} 2x - 5y = 3 \\ -4x + 3y = 8 \end{cases}$$

the coefficient matrix, D, D_x, and D_y are as follows.

Coefficient Matrix	D	D_x	D_y
$\begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$	$\begin{vmatrix} 2 & -5 \\ -4 & 3 \end{vmatrix}$	$\begin{vmatrix} 3 & -5 \\ 8 & 3 \end{vmatrix}$	$\begin{vmatrix} 2 & 3 \\ -4 & 8 \end{vmatrix}$