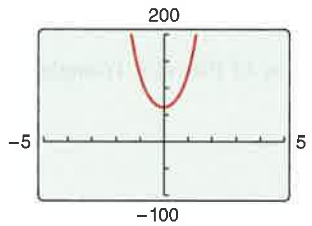
▷ **TECHNOLOGY** Use a graphing utility to check the expansion in Example 6. Graph the original binomial expression and the expansion in the same viewing window. The graphs should coincide, as shown in the figure below.




### EXAMPLE 6 Expanding a Binomial

Write the expansion of $(x^2 + 4)^3$.

**Solution** Use the third row of Pascal's Triangle.

$$(x^2 + 4)^3 = (1)(x^2)^3 + (3)(x^2)^2(4) + (3)x^2(4^2) + (1)(4^3)$$
$$= x^6 + 12x^4 + 48x^2 + 64$$

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Write the expansion of $(5 + y^2)^3$.

Sometimes you will need to find a specific term in a binomial expansion. Instead of writing the entire expansion, use the fact that, from the Binomial Theorem, the $(r + 1)$th term is ${}_nC_r x^{n-r} y^r$.

### EXAMPLE 7 Finding a Term or Coefficient

a. Find the sixth term of $(a + 2b)^8$.

b. Find the coefficient of the term $a^6b^5$ in the expansion of $(3a - 2b)^{11}$.

**Solution**

a. Remember that the formula is for the $(r + 1)$th term, so $r$ is one less than the number of the term you need. So, to find the sixth term in this binomial expansion, use $r = 5$, $n = 8$, $x = a$, and $y = 2b$.

$$ {}_nC_r x^{n-r} y^r = {}_8C_5 a^3(2b)^5$$
$$= 56a^3(32b^5)$$
$$= 1792a^3b^5$$

b. In this case, $n = 11$, $r = 5$, $x = 3a$, and $y = -2b$. Substitute these values to obtain

$$ {}_nC_r x^{n-r} y^r = {}_{11}C_5(3a)^6(-2b)^5$$
$$= (462)(729a^6)(-32b^5)$$
$$= -10{,}777{,}536a^6b^5.$$

So, the coefficient is $-10{,}777{,}536$.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

a. Find the fifth term of $(a + 2b)^8$.

b. Find the coefficient of the term $a^4b^7$ in the expansion of $(3a - 2b)^{11}$.

**Summarize** (Section 9.5)

1. State the Binomial Theorem *(page 648)*. For examples of using the Binomial Theorem to find binomial coefficients, see Examples 1 and 2.
2. Explain how to use Pascal's Triangle to find binomial coefficients *(page 650)*. For an example of using Pascal's Triangle to find binomial coefficients, see Example 3.
3. Explain how to use binomial coefficients to write a binomial expansion *(page 651)*. For examples of using binomial coefficients to write binomial expansions, see Examples 4–6.



## 9.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

**Vocabulary: Fill in the blanks.**

1. When you find the terms that result from raising a binomial to a power, you are ________ the binomial.
2. The coefficients of a binomial expansion are called ________ ________.
3. To find binomial coefficients, you can use the ________ ________ or ________ ________.
4. The symbol used to denote a binomial coefficient is ________ or ________.

### Skills and Applications

**Finding a Binomial Coefficient** In Exercises 5–12, find the binomial coefficient.

5. ${}_5C_3$  6. ${}_7C_6$
7. ${}_{12}C_0$  8. ${}_{20}C_{20}$
9. $\binom{10}{4}$  10. $\binom{10}{6}$
11. $\binom{100}{98}$  12. $\binom{100}{2}$

**Using Pascal's Triangle** In Exercises 13–16, evaluate using Pascal's Triangle.

13. ${}_6C_3$  14. ${}_4C_2$
15. $\binom{5}{1}$  16. $\binom{7}{4}$

**Expanding a Binomial** In Exercises 17–24, use the Binomial Theorem to write the expansion of the expression.

17. $(x + 1)^6$  18. $(x + 1)^4$
19. $(y - 3)^3$  20. $(y - 2)^5$
21. $(r + 3s)^3$  22. $(x + 2y)^4$
23. $(3a - 4b)^5$  24. $(2x - 5y)^5$

**Expanding an Expression** In Exercises 25–38, expand the expression by using Pascal's Triangle to determine the coefficients.

25. $(a + 6)^4$  26. $(a + 5)^5$
27. $(y - 1)^6$  28. $(y - 4)^4$
29. $(3 - 2z)^4$  30. $(3v + 2)^6$
31. $(x + 2y)^5$  32. $(2t - s)^5$
33. $(x^2 + y^2)^4$  34. $(x^2 + y^2)^6$
35. $\left(\frac{1}{x} + y\right)^5$  36. $\left(\frac{1}{x} + 2y\right)^6$
37. $2(x - 3)^4 + 5(x - 3)^2$
38. $(4x - 1)^3 - 2(4x - 1)^4$

**Finding a Term** In Exercises 39–46, find the specified $n$th term in the expansion of the binomial.

39. $(x + y)^{10}$, $n = 4$  40. $(x - y)^6$, $n = 2$
41. $(x - 6y)^5$, $n = 3$  42. $(x + 2z)^7$, $n = 4$
43. $(4x + 3y)^9$, $n = 8$  44. $(5a + 6b)^5$, $n = 5$
45. $(10x - 3y)^{12}$, $n = 10$  46. $(7x + 2y)^{15}$, $n = 7$

**Finding a Coefficient** In Exercises 47–54, find the coefficient $a$ of the term in the expansion of the binomial.

| | Binomial | Term |
|---|---|---|
| 47. | $(x + 2)^6$ | $ax^3$ |
| 48. | $(x - 2)^6$ | $ax^3$ |
| 49. | $(4x - y)^{10}$ | $ax^2y^8$ |
| 50. | $(x - 2y)^{10}$ | $ax^8y^2$ |
| 51. | $(2x - 5y)^9$ | $ax^4y^5$ |
| 52. | $(3x + 4y)^8$ | $ax^6y^2$ |
| 53. | $(x^2 + y)^{10}$ | $ax^8y^6$ |
| 54. | $(z^2 - t)^{10}$ | $az^4t^8$ |

**Expanding an Expression** In Exercises 55–60, use the Binomial Theorem to write the expansion of the expression.

55. $(\sqrt{x} + 5)^3$  56. $(2\sqrt{t} - 1)^3$
57. $(x^{2/3} - y^{1/3})^3$  58. $(u^{3/5} + 2)^5$
59. $(3\sqrt{t} + \sqrt[4]{t})^4$  60. $(x^{3/4} - 2x^{5/4})^4$

**Simplifying a Difference Quotient** In Exercises 61–66, simplify the difference quotient, using the Binomial Theorem if necessary.

$$\frac{f(x + h) - f(x)}{h} \quad \text{Difference quotient}$$

61. $f(x) = x^3$  62. $f(x) = x^4$
63. $f(x) = x^6$  64. $f(x) = x^7$
65. $f(x) = \sqrt{x}$  66. $f(x) = \frac{1}{x}$

**Expanding a Complex Number** **In Exercises 67–72, use the Binomial Theorem to expand the complex number. Simplify your result.**

**67.** $(1 + i)^4$ **68.** $(2 - i)^5$

**69.** $(2 - 3i)^6$ **70.** $(5 + \sqrt{-9})^3$

**71.** $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3$ **72.** $(5 - \sqrt{3}i)^4$

**Approximation** **In Exercises 73–76, use the Binomial Theorem to approximate the quantity accurate to three decimal places. For example, in Exercise 73, use the expansion**

$$(1.02)^8 = (1 + 0.02)^8$$
$$= 1 + 8(0.02) + 28(0.02)^2 + \cdot\cdot\cdot + (0.02)^8.$$

**73.** $(1.02)^8$ **74.** $(2.005)^{10}$

**75.** $(2.99)^{12}$ **76.** $(1.98)^9$

**Probability** **In Exercises 77–80, consider $n$ independent trials of an experiment in which each trial has two possible outcomes: "success" or "failure." The probability of a success on each trial is $p$, and the probability of a failure is $q = 1 - p$. In this context, the term ${}_nC_k p^k q^{n-k}$ in the expansion of $(p + q)^n$ gives the probability of $k$ successes in the $n$ trials of the experiment.**

**77.** You toss a fair coin seven times. To find the probability of obtaining four heads, evaluate the term

$${}_7C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^3$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^7$.

**78.** The probability of a baseball player getting a hit during any given time at bat is $\frac{1}{4}$. To find the probability that the player gets three hits during the next 10 times at bat, evaluate the term

$${}_{10}C_3\left(\tfrac{1}{4}\right)^3\left(\tfrac{3}{4}\right)^7$$

in the expansion of $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$.

**79.** The probability of a sales representative making a sale with any one customer is $\frac{1}{3}$. The sales representative makes eight contacts a day. To find the probability of making four sales, evaluate the term

$${}_8C_4\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)^4$$

in the expansion of $\left(\frac{1}{3} + \frac{2}{3}\right)^8$.

**80.** To find the probability that the sales representative in Exercise 79 makes four sales when the probability of a sale with any one customer is $\frac{1}{2}$, evaluate the term

$${}_8C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^4$$

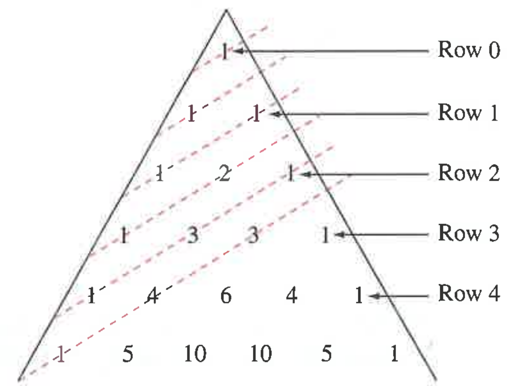
in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^8$.

**Graphical Reasoning** **In Exercises 81 and 82, use a graphing utility to graph $f$ and $g$ in the same viewing window. What is the relationship between the two graphs? Use the Binomial Theorem to write the polynomial function $g$ in standard form.**

**81.** $f(x) = x^3 - 4x$
$g(x) = f(x + 4)$

**82.** $f(x) = -x^4 + 4x^2 - 1$
$g(x) = f(x - 3)$

**83. Finding a Pattern** Describe the pattern formed by the sums of the numbers along the diagonal line segments shown in Pascal's Triangle (see figure).




**84. Error Analysis** Describe the error.

$$(x - 3)^3 = {}_3C_0x^3 + {}_3C_1x^2(3) + {}_3C_2x(3)^2 + {}_3C_3(3)^3$$
$$= 1x^3 + 3x^2(3) + 3x(3)^2 + 1(3)^3$$
$$= x^3 + 9x^2 + 27x + 27$$

**85. Child Support** The amounts $f(t)$ (in billions of dollars) of child support collected in the United States from 2005 through 2014 can be approximated by the model

$$f(t) = -0.056t^2 + 1.62t + 16.4, \quad 5 \le t \le 14$$

where $t$ represents the year, with $t = 5$ corresponding to 2005. *(Source: U.S. Department of Health and Human Services)*

(a) You want to adjust the model so that $t = 5$ corresponds to 2010 rather than 2005. To do this, you shift the graph of $f$ five units *to the left* to obtain $g(t) = f(t + 5)$. Use binomial coefficients to write $g(t)$ in standard form.

(b) Use a graphing utility to graph $f$ and $g$ in the same viewing window.

(c) Use the graphs to estimate when the child support collections exceeded \$27 billion.



**86. Electricity** The table shows the average prices $f(t)$ (in cents per kilowatt-hour) of residential electricity in the United States from 2007 through 2014. *(Source: U.S. Energy Information Administration)*

DATA

Spreadsheet at LarsonPrecalculus.com

| Year | Average Price, $f(t)$ |
|---|---|
| 2007 | 10.65 |
| 2008 | 11.26 |
| 2009 | 11.51 |
| 2010 | 11.54 |
| 2011 | 11.72 |
| 2012 | 11.88 |
| 2013 | 12.13 |
| 2014 | 12.52 |

(a) Use the *regression* feature of a graphing utility to find a cubic model for the data. Let $t$ represent the year, with $t = 7$ corresponding to 2007.

(b) Use the graphing utility to plot the data and the model in the same viewing window.

(c) You want to adjust the model so that $t = 7$ corresponds to 2012 rather than 2007. To do this, you shift the graph of $f$ five units *to the left* to obtain $g(t) = f(t + 5)$. Use binomial coefficients to write $g(t)$ in standard form.

(d) Use the graphing utility to graph $g$ in the same viewing window as $f$.

(e) Use both models to predict the average price in 2015. Do you obtain the same answer?

(f) Do your answers to part (e) seem reasonable? Explain.

(g) What factors do you think contributed to the change in the average price?

## Exploration

**True or False?** **In Exercises 87 and 88, determine whether the statement is true or false. Justify your answer.**

**87.** The Binomial Theorem could be used to produce each row of Pascal's Triangle.

**88.** A binomial that represents a difference cannot always be accurately expanded using the Binomial Theorem.

**89. Writing** Explain how to form the rows of Pascal's Triangle.

**90. Forming Rows of Pascal's Triangle** Form rows 8–10 of Pascal's Triangle.

**91. Graphical Reasoning** Use a graphing utility to graph the functions in the same viewing window. Which two functions have identical graphs, and why?

$f(x) = (1 - x)^3$
$g(x) = 1 - x^3$
$h(x) = 1 + 3x + 3x^2 + x^3$
$k(x) = 1 - 3x + 3x^2 - x^3$
$p(x) = 1 + 3x - 3x^2 + x^3$

**92. HOW DO YOU SEE IT?** The expansions of $(x + y)^4$, $(x + y)^5$, and $(x + y)^6$ are shown below.

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$
$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$
$$(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$

(a) Explain how the exponent of a binomial is related to the number of terms in its expansion.

(b) How many terms are in the expansion of $(x + y)^n$?

**Proof** **In Exercises 93–96, prove the property for all integers $r$ and $n$, where $0 \le r \le n$.**

**93.** ${}_nC_r = {}_nC_{n-r}$

**94.** ${}_nC_0 - {}_nC_1 + {}_nC_2 - \cdot\cdot\cdot \pm {}_nC_n = 0$

**95.** ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$

**96.** The sum of the numbers in the $n$th row of Pascal's Triangle is $2^n$.

**97. Binomial Coefficients and Pascal's Triangle** Complete the table. What characteristic of Pascal's Triangle does this table illustrate?

| $n$ | $r$ | ${}_nC_r$ | ${}_nC_{n-r}$ |
|---|---|---|---|
| 9 | 5 | | |
| 7 | 1 | | |
| 12 | 4 | | |
| 6 | 0 | | |
| 10 | 7 | | |